

Lecture 07

Heat eq with Neumann boundary cond.

$$\left. \begin{aligned} u_t &= u_{xx} & x \in (0,1), \quad t > 0 \\ u_x(0,t) &= u_x(1,t) = 0 & t > 0 \\ u(x,0) &= f(x) & x \in [0,1] \end{aligned} \right\} (1)$$

Guess solution form

$$u_k(x,t) = X_k(x) T_k(t)$$

$$\text{Boundary cond: } X_k'(0) = X_k'(1) = 0$$

$$(u_k)_t = (u_k)_{xx} \Rightarrow \frac{T_k'(t)}{T_k(t)} = \frac{X_k''(x)}{X_k(x)} = -\lambda_k \quad (2)$$

for some $\lambda_k \in \mathbb{R}$.

Eigenvalue problem:

$$(2) \Rightarrow \left. \begin{aligned} -X_k'' &= \lambda_k X_k & x \in (0,1) \\ X_k'(0) &= X_k'(1) = 0 \end{aligned} \right\} (3)$$

The properties

$$\begin{aligned}\langle -\underline{x}_k'', \underline{x}_k \rangle &= -\int_0^1 \underline{x}_k''(x) \underline{x}(x) dx \\ &= \left[-\underline{x}_k' \underline{x}_k \right]_0^1 + \int_0^1 (\underline{x}_k'(x))^2 dx \\ &= \langle \underline{x}_k', \underline{x}_k' \rangle.\end{aligned}$$

and

$$-\underline{x}_k'' = \lambda_k \underline{x}_k \quad | \quad \langle \cdot | \underline{x}_k \rangle$$

$$\Rightarrow \lambda_k = \frac{\langle \underline{x}_k', \underline{x}_k' \rangle}{\langle \underline{x}_k, \underline{x}_k \rangle} \geq 0.$$

Writing $\beta_k = \sqrt{\lambda_k}$, problem (3) becomes

$$\underline{x}_k''(x) = -\beta_k^2 \underline{x}_k(x)$$

General solution :

$$\underline{x}_k(x) = C_1 \cos(\beta_k x) + C_2 \sin(\beta_k x)$$

$$\text{with } \underline{x}_k'(x) = \beta_k (C_2 \cos(\beta_k x) - C_1 \sin(\beta_k x))$$

$$X_k'(0) = 0 \Rightarrow c_2 = 0$$

$$X_k'(1) = 0 \Rightarrow B_k \sin(B_k) = 0$$

$$\Rightarrow B_k = k\pi, \quad k=0,1,2,\dots$$

Setting $c_1 = 1$, we obtain

$$X_k(x) = \cos(k\pi x).$$

Eq for T_k from (2):

$$T_k' = -\lambda_k T_k \quad t \geq 0$$

$$\Rightarrow T_k(t) = e^{-\lambda_k t} \underbrace{T_k(0)}_{\text{set}=1} = e^{-(k\pi)^2 t}$$

for $k=0,1,\dots$

Family of solutions to (1):

$$u_k(x,t) = e^{-(k\pi)^2 t} \cos(k\pi x) \quad k=0,1,\dots$$

and by linearity,

$$u(x,t) = \sum_{k=0}^N c_k e^{-(k\pi)^2 t} \cos(k\pi x)$$

is also a solution.

Formal solutions

$$\text{If } f(x) = \sum_{k=0}^{\infty} c_k \cos(k\pi x),$$

$$\text{then } u(x,t) = \sum_{k=0}^{\infty} c_k e^{-(k\pi)^2 t} \cos(k\pi x) \quad (4)$$

is a formal solution to (1).

Fourier cosine series

For $v_k(x) := \cos(k\pi x)$ it holds that

$$\langle v_k, v_m \rangle = \begin{cases} 0 & k \neq m \\ \frac{1}{2} & k = m \geq 1 \\ 1 & k = m = 0 \end{cases} \quad (5)$$

(Exercise)

Fourier cosine series:

$$f(x) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k \cos(k\pi x) \quad | \langle \cdot, v_m \rangle$$

Determine c_m by:

$$\langle f, v_m \rangle = \frac{c_0}{2} \langle \underbrace{1}_{=v_0} | v_m \rangle + \sum_{k=1}^{\infty} c_k \langle v_k, v_m \rangle$$

$$= \frac{c_m}{2}$$

$$\Rightarrow c_m = 2 \langle f, \cos(m\pi x) \rangle$$

$$m = 0, 1, \dots,$$

Example

$f = 1 = V_0(x)$ has coefficients

$$C_m = 2 \langle f, V_m \rangle = 2 \langle V_0, V_m \rangle$$

$$\stackrel{(5)}{=} \begin{cases} 2 & \text{if } m=0 \\ 0 & \text{otherwise} \end{cases}$$

(Cosine series $1 = \frac{C_0}{2} V_0(x) = 1$.)

Example

$$f(x) = x$$

$$C_k = 2 \langle f, V_k(x) \rangle = 2 \int_0^1 x \cos(k\pi x) dx$$

for $k=0$

$$C_0 = 2 \int_0^1 x dx = 1$$

for $k > 0$

$$C_k = 2 \int_0^1 x \cos(k\pi x) dx$$

$$= 2 \left[\frac{x \sin(k\pi x)}{k\pi} \right]_0^1 - \frac{2}{k\pi} \int_0^1 \sin(k\pi x) dx$$

$$= \frac{2}{(k\pi)^2} \left[\cos(k\pi x) \right]_0^1$$

$$= \frac{2}{(k\pi)^2} ((-1)^k - 1)$$

$$= \begin{cases} -\frac{4}{(k\pi)^2} & \text{if } k = 1, 3, 5, \dots \\ 0 & \text{if } k = 2, 4, 6, \dots \end{cases}$$

Cosine series:

$$x = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos((2k-1)\pi x)$$

And eq

$$u_t = u_{xx}$$

$$t > 0, x \in (0,1)$$

$$u_x(0,t) = u_x(1,t)$$

$$t > 0$$

$$u(x,0) = x$$

has formal solution (see (4))

$$u(x,t) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} e^{-(2k-1)^2 \pi^2 t} \cos((2k-1)\pi x)$$