

Lecture 8

Consider

$$\left. \begin{aligned} u_t &= u_{xx} & x \in (0,1) \text{ , } t > 0 \\ u(0,t) &= u(1,t) = 0 & t > 0 \end{aligned} \right\} (1)$$

$$u(x,0) = f(x) \quad x \in (0,1) \quad (2)$$

Assume that $u(x,t)$ solves (1)-(2) and
that $u \in C^{2,1}([0,1] \times [0,\infty))$

$$C^{2,1} = \{ u \in C([0,1] \times [0,\infty)) \mid u_t, u_{xx} \in C([0,1] \times [0,\infty)) \}$$

Energy argument

$$\text{Let } E(t) := \int_0^1 u^2(x,t) dx \quad t \geq 0$$

and consider :

$$E'(t) = \frac{d}{dt} \int_0^1 u^2(x,t) dx$$

$$= \int_0^1 \frac{\partial}{\partial t} u^2(x,t) dx$$

will justify $\uparrow$ interchange of $\frac{d}{dt}$ and $\int_0^1$ later

$$\begin{aligned}
&= 2 \int_0^1 u(x,t) u_t(x,t) dx \\
&\stackrel{(1)}{=} 2 \int_0^1 u u_{xx} dx \\
&= 2 \int_0^1 \frac{\partial}{\partial x} (u u_x) - (u_x)^2 dx \\
&= 2 \underbrace{\left[u(x,t) u_x(x,t) \right]_0^1}_{=0} - 2 \underbrace{\int_0^1 (u_x(x,t))^2 dx}_{\leq 0} \\
&\leq 0
\end{aligned}$$

Energy estimate:

$$E(t) \leq E(0) \quad \forall t \geq 0$$

or equivalently

$$\int_0^1 u^2(x,t) dx \leq \int_0^1 f^2(x) dx \quad \forall t \geq 0$$

Theorem

Let u and v solve (1)
with $u(x,0) = f(x)$ and $v(x,0) = g(x)$,
and assume that
 $u, v \in C^{2,1}([0,1] \times [0,\infty))$.

Then for all $t \geq 0$

$$\int_0^1 (u(x,t) - v(x,t))^2 dx \leq \int_0^1 (f(x) - g(x))^2 dx \quad (3)$$

Proof: Since PDE is linear,

$$w := u - v$$

solves heat eq.

$$w_t = w_{xx} \quad x \in (0,1), \quad t > 0$$

$$w(0,t) = w(1,t) = 0 \quad t > 0$$

$$w(x,0) = f(x) - g(x)$$

and $w, w_t, w_{xx} \in C([0,1] \times [0,\infty))$.

We obtain (3) by previous energy argument. $\square$

Observe if $g = f$ in (3), then

$$\int_0^1 (u(t, x) - v(t, x))^2 dx \leq 0$$

$$\Rightarrow u(t, x) = v(t, x).$$

So for each f , the heat eq (1)-(2) has at most one solution in $C^{2,1}$.

Differentiation of integrals

Let g be a continuous function on $[a, b] \times [c, d]$, and assume that $g_y \in C([a, b] \times [c, d])$.

$[a, b] \times [c, d]$ is compact, so g_y is uniformly continuous on $[a, b] \times [c, d]$.

In particular, this means that

for any $\varepsilon > 0$, $\exists h > 0$ such that

$$\sup_{\delta \leq h} \sup_{x \in [a, b]} |g_y(x, y + \delta) - g_y(x, y)| < \varepsilon$$

for any $y \in (c, d)$.

Implication:

$$\lim_{h \rightarrow 0} \sup_{\delta \leq h} \sup_{x \in [a, b]} |g_y(x, y + \delta) - g_y(x, y)| = 0 \quad (4)$$

for any $y \in (c, d)$.

Theorem

If $g, g_y \in C([a, b] \times [c, d])$, then the function

$$G(y) := \int_a^b g(x, y) \, dx \quad (5)$$

is differentiable and

$$G'(y) = \int_a^b g_y(x, y) \, dx$$

Proof

Need to show that for any $y \in (c, d)$,

$$\lim_{h \rightarrow 0} \frac{G(y+h) - G(y)}{h} = \int_a^b g_y(x, y) \, dx.$$

Let $h > 0$ and $y \in (c, d)$. Then

$$\begin{aligned} \frac{G(y+h) - G(y)}{h} &= \int_a^b \frac{g(x, y+h) - g(x, y)}{h} \, dx \\ &= \int_a^b g_y(x, y + \theta) \, dx \end{aligned}$$

for some $\delta(h) \in [0, h]$ by
the mean-value theorem.

Consequently,

$$\left| \frac{G(y+h) - G(y)}{h} - \int_a^b g_y(x, y) dy \right|$$

$$= \left| \int_a^b g_y(x, y+\delta) - g_y(x, y) dy \right|$$

$$\leq \int_a^b |g_y(x, y+\delta) - g_y(x, y)| dy$$

$$\leq (b-a) \sup_{\delta \leq h} \sup_{x \in [a, b]} |g_y(x, y+\delta) - g_y(x, y)|$$

Take limit $h \downarrow 0$:

$$\lim_{h \downarrow 0} \left| \frac{G(y+h) - G(y)}{h} - \int_a^b g_y(x, y) dy \right|$$

$$\leq (b-a) \times \lim_{h \downarrow 0} \sup_{\delta \leq h} \sup_{x \in [a, b]} |g(x, y+\delta) - g(x, y)|$$

$$\stackrel{(4)}{=} 0.$$



Result applies to solution of heat eq (1)-(2) with $u \in C^{2,1}([0,1] \times [0, \infty))$ as follows:

- For any $T > 0$,

$$u, u_t \in C([0,1] \times [0,T]).$$

- This implies that

$$u^2, \underbrace{(u^2)_t}_{2uu_t} \in C([0,1] \times [0,T]).$$

- For $E(t) = \int_0^1 u^2(x,t) dx$, previous theorem implies that

$$E'(t) = \int_0^1 (u^2)_t(x,t) dx \quad (6)$$

holds for all $t \in (0,T)$

- Since $T > 0$ is arbitrary,

(6) holds for any $t > 0$.

Energy argument for nonlinear PDE

$$\left. \begin{aligned} u_t &= u_{xx} - u^3 & x \in (0,1), t > 0 \\ u(0,t) &= u(1,t) = 0 & t > 0 \end{aligned} \right\} (7)$$

$$u(x,0) = f(x) \quad (8)$$

Assume $u \in C^{2,1}([0,1] \times [0, \infty))$
solves (7)-(8) and let

$$E(t) := \int_0^1 u^2(x,t) dx.$$

Then

$$E'(t) = \frac{d}{dt} \int_0^1 u^2(x,t) dx$$

$$= \int_0^1 \frac{\partial}{\partial t} u^2 dx$$

$$= 2 \int_0^1 u u_t dx$$

$$= 2 \int_0^1 u (u_{xx} - u^3) dx$$

$$= 2 \underbrace{[u u_x]_0^1}_{=0} - 2 \int_0^1 (u_x(x,t))^2 - (u(x,t))^4 dx$$

$$\leq 0$$

Gives growth bound for solution:

$$E(t) \leq E(0) \quad \forall t \geq 0$$

$$\int_0^1 u^2(x,t) dx \leq \int_0^1 f^2(x) dx$$

Uniqueness

Let u & v be solutions
of (7) with
 $u(x,0) = f(x)$ and $v(x,0) = g(x)$.

If $u, v \in C^{2,1}$, then $(u-v)$ solves

$$(u-v)_t = (u-v)_{xx} - (u^3 - v^3)$$

$$(u-v)(0,t) = (u-v)(1,t) \quad t > 0$$

$$(u-v)(x,0) = f(x) - g(x)$$

Consider energy

$$E(t) := \int_0^1 (u-v)^2(x,t) dx$$

Since $(u-v)^2 \in C^{2,1}$,

$$E'(t) = \int_0^1 \frac{\partial}{\partial t} (u-v)^2 dx$$

$$= 2 \int_0^1 (u-v)(u-v)_t dx$$

$$= 2 \int_0^1 (u-v)(u-v)_{xx} - (u-v)(u^3 - v^3) dx$$

$$= 2 \underbrace{[(u-v)(u-v)_x]_0^1}_{=0} - 2 \underbrace{\int_0^1 ((u-v)_x(x,t))^2 dx}_{\leq 0}$$

$$- 2 \underbrace{\int_0^1 (u-v)(u^3 - v^3) dx}_{\leq 0} \leq 0$$

Last integral is ≤ 0 since

$$(u-v)(u^3-v^3) = \begin{cases} \geq 0 & \text{if } u \geq v \\ \leq 0 & \text{if } u \leq v \end{cases}.$$

Conclusion:

$$\int_0^1 (u(x,t) - v(x,t))^2 dx \leq \int_0^1 (f(x) - g(x))^2 dx.$$

and (7)-(8) has at most one solution in $C^{2,1}$.