

Lecture 09

Consider

$$\left. \begin{array}{l} u_t = u_{xx} \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = f(x) \end{array} \quad \begin{array}{l} t > 0, x \in (0,1) \\ t \geq 0 \\ x \in (0,1) \end{array} \right\} (1)$$

Numerical method

$$\Delta x = \frac{1}{n+1} \quad \text{and} \quad \Delta t > 0,$$

Mesh (x_j, t_m) with

$$x_j = j \Delta x \quad j = 0, 1, \dots, n+1$$

$$t_m = m \Delta t \quad m \geq 0.$$

Approximations

$$u_t(x, t) \approx \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t}$$

and

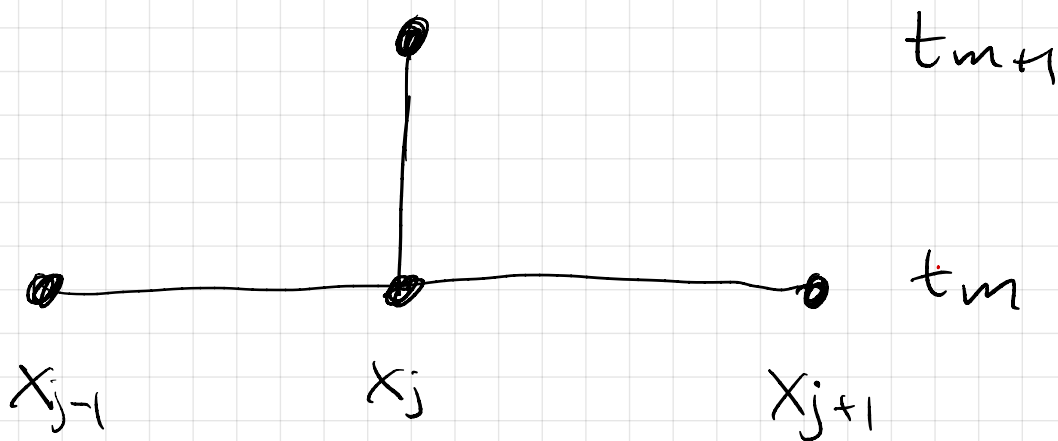
$$u_{xx}(x, t) \approx \frac{u(x - \Delta x, t) - 2u(x, t) + u(x + \Delta x, t)}{\Delta x^2}$$

motivate scheme for $v_j^m \approx u(x_j, t_m)$:

$$\frac{V_j^{m+1} - V_j^m}{\Delta t} = \frac{V_{j-1}^m - 2V_j^m + V_{j+1}^m}{\Delta x^2}$$

$$\left. \begin{array}{l} \text{for } j=1, \dots, n \text{ and } m \geq 0, \\ \text{with } V_0^m = V_{n+1}^m = 0 \quad m \geq 0 \\ \text{and } V_j^0 = f(x_j) \quad j=1, \dots, n. \end{array} \right\} (2)$$

Numerical stencil



(V_j^{m+1} is computed from V_{j-1}^m , V_j^m and V_{j+1}^m)

Introducing

$$A = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \\ & & -1 & 2 \end{bmatrix} \in \mathbb{R}^{n \times n} \quad \& \quad v^m = \begin{bmatrix} v_1^m \\ \vdots \\ v_n^m \end{bmatrix},$$

we rewrite (2)

$$\begin{aligned} v^{m+1} &= \left(I - \frac{\Delta t}{\Delta x^2} A \right) v^m \quad m \geq 0 \quad (3) \\ &= \left(I - \frac{\Delta t}{\Delta x^2} A \right)^m v^0. \end{aligned}$$

$\frac{1}{\Delta x^2} A$ has eigenpairs (μ_k, w_k)

$$\mu_k = \frac{4}{\Delta x^2} \sin^2\left(\frac{k\pi \Delta x}{2}\right)$$

$$w_{k,j} = w_k(x_j) = \sin(k\pi x_j) \quad j=1, \dots, n$$

and $\{w_k\}_{k=1}^n$ is an orthogonal basis for $\mathbb{R}^n$. (Lecture 5)

Since

$$\left(I - \frac{\Delta t}{\Delta x^2} A\right)^m w_k = (1 - \Delta t \mu_k)^m w_k$$

$$\bar{u}_k^m := (1 - \Delta t \mu_k)^m w_k \in \mathbb{R}^n \text{ with}$$

$$\bar{u}_{k,j}^m = (1 - \Delta t \mu_k)^m \sin(k\pi x_j)$$

are particular solutions of (3) for $k=1, \dots, n$.

Verification:

$$\begin{aligned} \bar{u}_k^{m+1} &= (1 - \Delta t \mu_k)^{m+1} w_k \\ &= \left(I - \frac{\Delta t}{\Delta x^2} A\right) \underbrace{(1 - \Delta t \mu_k)^m w_k}_{\bar{u}_k^m} \end{aligned}$$

Linearity of (3) gives general solution

$$\begin{aligned} v_j^m &= \sum_{k=1}^n c_k \bar{u}_k^m \\ &= \sum_{k=1}^n c_k (1 - \Delta t \mu_k)^m \sin(k\pi x_j) \quad (4) \end{aligned}$$

where

$$v_j^0 = f(x_j) = \sum_{k=1}^n c_k \sin(k\pi x_j)$$

$\Rightarrow$

$$c_k = 2 \langle f, \sin(k\pi x) \rangle_{\Delta x} \quad (5)$$
$$= 2\Delta x \sum_{j=1}^n f(x_j) \sin(k\pi x_j)$$

Compare (4) & (5) to formal continuous solution of (1):

$$u(x_j, t_m) = \sum_{k=1}^{\infty} d_k e^{-(k\pi)^2 t_m} \sin(k\pi x_j)$$

with $d_k = 2 \int_0^1 f(x) \sin(k\pi x) dx$.

Convergence

If $\frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$ and f smooth, then

$$\lim_{\substack{\Delta t \rightarrow 0 \\ \Delta x \rightarrow 0}} V_j^m = u(x_j, t_m)$$

(see TW 4.2.4).

Von Neumann Stability

Recall that PDE

$$u_t = u_{xx} \quad x \in (0,1), \quad t > 0$$

has particular solutions

- $u_k(x,t) = e^{-(k\pi)^2 t} \sin(k\pi x) \quad k=1,2,\dots$
when BC are Dirichlet
 $u(0,t) = u(1,t) = 0 \quad t \geq 0,$

- $u_k(x,t) = e^{-(k\pi)^2 t} \cos(k\pi x) \quad k=0,1,\dots$
when BC are Neumann
 $u_x(0,t) = u_x(1,t) = 0 \quad t \geq 0$

Observation: all $u_k(x,t)$ above
are linear combinations of

$$u_k(x,t) = e^{-(k\pi)^2 t} e^{ik\pi x} \quad k \in \mathbb{Z},$$

(since $\sin(y) = \frac{e^{iy} - e^{-iy}}{2i}$ & $\cos(y) = \frac{e^{iy} + e^{-iy}}{2}$)

Extension :

For

$$C_0 u_{tt} + C_1 u_t = C_2 u_{xx} + C_3 u_x + C_4 u \quad t > 0, x \in (0,1)$$

with constant $C_0, \dots, C_4 \in \mathbb{R}$,
particular solutions are linear comb.
of

$$u_k(x, t) = T_k(t) e^{ik\pi x} \quad k \in \mathbb{Z}$$

(where T_k depends on problem)

Correspondingly, discrete particular
solutions are linear combinations of

$$\bar{u}_k(x_j, t_m) = (a_k)^m e^{ik\pi x_j} \quad k \in \mathbb{Z}$$

(where $a_k \in \mathbb{C}$ depends on the
problem & scheme).

Note that

$$|\bar{U}_k(x_j, t_m)| = |a_k|^m \quad \text{and}$$

$$|U_k(x_j, t_m)| = |T(t_m)|.$$

Reasonable growth condition
for numerical solution:

A scheme is stable in the
sense of Von Neumann if

$$\sup_{k \in \mathbb{Z}} |a_k|^m \leq \sup_{k \in \mathbb{Z}} |T_k(t_m)|$$

holds for all $m \geq 0$

How to use condition:

- ① given PDE, find T_k and a_k for discrete problem $\forall k \in \mathbb{Z}$
- ② Determine Δt & Δx so that Neumann stability holds, if possible.

Example: For

$$u_t = u_{xx} \quad x \in (0,1), t > 0,$$

inserting $u_k(x,t) = T_k(t) e^{ik\pi x}$ gives

$$T_k'(t) e^{ik\pi x} = -(k\pi)^2 T_k(t) e^{ik\pi x}$$

$$\Rightarrow T_k(t) = e^{-\underbrace{(k\pi)^2 t}_{:=1}} \underbrace{T_k(0)}_{:=1} = e^{-(k\pi)^2 t}$$

Associated discrete problem:

$$\frac{V_j^{m+1} - V_j^m}{\Delta t} = \frac{V_{j-1}^m - 2V_j^m + V_{j+1}^m}{\Delta x^2}$$

Inserting $\bar{u}_{kj}^m = (a_k)^m e^{ik\pi x_j}$ above

Yields

$$\frac{(a_k - 1) \cancel{(a_k)^m} \cancel{e^{ik\pi x_j}}}{\Delta t} = \frac{e^{-ik\pi \Delta x} - 2 + e^{ik\pi \Delta x}}{\Delta x^2} \cancel{(a_k)^m} \cancel{e^{ik\pi x_j}}$$

$$\Rightarrow a_k = 1 - 2 \frac{\Delta t}{\Delta x^2} (1 - \cos(k\pi \Delta x))$$

$$= 1 - 4 \frac{\Delta t}{\Delta x^2} \sin^2\left(\frac{k\pi \Delta x}{2}\right)$$

Since

$$\sup_{k \in \mathbb{Z}} |T_k(t_m)| = \sup_{k \in \mathbb{Z}} e^{-(k\pi)^2 t_m} = 1,$$

the scheme is Neumann stable
iff

$$|a_k|^m \leq 1 \quad \forall m \geq 0, k \in \mathbb{Z}$$

$$\Leftrightarrow |a_k| = \left| 1 - 4 \frac{\Delta t}{\Delta x^2} \sin^2\left(\frac{k\pi\Delta x}{2}\right) \right| \leq 1 \quad \forall k \in \mathbb{Z}$$

$$\Leftrightarrow \underbrace{\frac{\Delta t}{\Delta x^2} \sin^2\left(\frac{k\pi\Delta x}{2}\right)} \leq \frac{1}{2} \quad \forall k \in \mathbb{Z}$$

$$\Leftrightarrow \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2} \quad (8)$$

(since $\sin\left(\frac{k\pi\Delta x}{2}\right) = 1$ for $k = n+1 = \frac{1}{\Delta x}$)

(8) is called a CFL condition
for the explicit method (7).

When $\frac{\Delta t}{\Delta x^2} > \frac{1}{2}$ solutions of (7)
may explode or oscillate wildly.

Example Consider PDE

$$u_t = u_{xx} + u \quad x \in (0,1) \quad t > 0$$

and explicit scheme

$$\frac{V_j^{m+1} - V_j^m}{\Delta t} = \frac{V_{j-1}^m - 2V_j^m + V_{j+1}^m}{\Delta x^2} + V_j^m$$

Inserting $u_k(x,t) = T_k(t) e^{ik\pi x}$ into PDE:

$$T_k'(t) e^{ik\pi x} = (-(k\pi)^2 + 1) T_k(t) e^{ik\pi x}$$

$$\Rightarrow T_k(t) = e^{(1-(k\pi)^2)t} \underbrace{T_k(0)}_{=1}$$

$$= e^{(1-(k\pi)^2)t}$$

$$\Rightarrow \sup_{k \in \mathbb{Z}} |T_k(t)| \leq e^t$$

Inserting $\bar{u}_{kj}^m = (a_k)^m e^{ik\pi x_j}$ in

scheme:

$$\frac{(a_k - 1)(a_k)^m e^{ik\pi x_j}}{\Delta t} = \left(\frac{e^{-ik\pi \Delta x} - 2 + e^{ik\pi \Delta x}}{\Delta x^2} + 1 \right) (a_k)^m e^{ik\pi x_j}$$

$\Rightarrow$

$$a_k = 1 + \Delta t - \frac{2\Delta t}{\Delta x^2} \left(1 - \cos(k\pi\Delta x) \right)$$

$$= 1 + \Delta t - \frac{4\Delta t}{\Delta x^2} \sin^2\left(\frac{k\pi\Delta x}{2}\right)$$

Scheme is Neumann stable if

$$\sup_{k \in \mathbb{Z}} |a_k|^m \leq \sup_{k \in \mathbb{Z}} |T_k(t_m)| = e^{t_m} = e^{m\Delta t}$$

for all $m \geq 0$.

Equivalent to

$$\sup_{k \in \mathbb{Z}} |a_k| \leq e^{\Delta t}$$

$$\Leftrightarrow -e^{\Delta t} \leq 1 + \Delta t - \frac{4\Delta t}{\Delta x^2} \sin^2\left(\frac{k\pi\Delta x}{2}\right) \leq e^{\Delta t} \quad \forall k$$

$\uparrow$
always holds

$$\Leftrightarrow \frac{4\Delta t}{\Delta x^2} \sin^2\left(\frac{k\pi\Delta x}{2}\right) \leq 1 + \Delta t + e^{\Delta t} \quad \forall k$$

$(k = n+1 = \Delta x^{-1})$

$$\Leftrightarrow \frac{\Delta t}{\Delta x^2} \leq \frac{1 + \Delta t + e^{\Delta t}}{4} \quad \text{(CFL)}$$

(TW 4.3.2 derives the sufficient condition

$$\frac{\Delta t}{\Delta x^2} \leq \frac{1}{2},$$

which is more restrictive than
(CFL) on previous page.)