

Lecture 10

Implicit method for heat eq.

$$\left. \begin{aligned} u_t &= u_{xx} & x \in (0,1), t > 0 \\ u(0,t) &= u(1,t) = 0 & t \geq 0 \end{aligned} \right\} (1)$$

$$u(x,0) = f(x) \quad x \in (0,1) \quad (2)$$

Approximations

$$u_t(x, t + \Delta t) \approx \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t}$$

$$u_{xx}(x, t + \Delta t) \approx \frac{u(x - \Delta x, t + \Delta t) - 2u(x, t + \Delta t) + u(x + \Delta x, t + \Delta t)}{\Delta x^2}$$

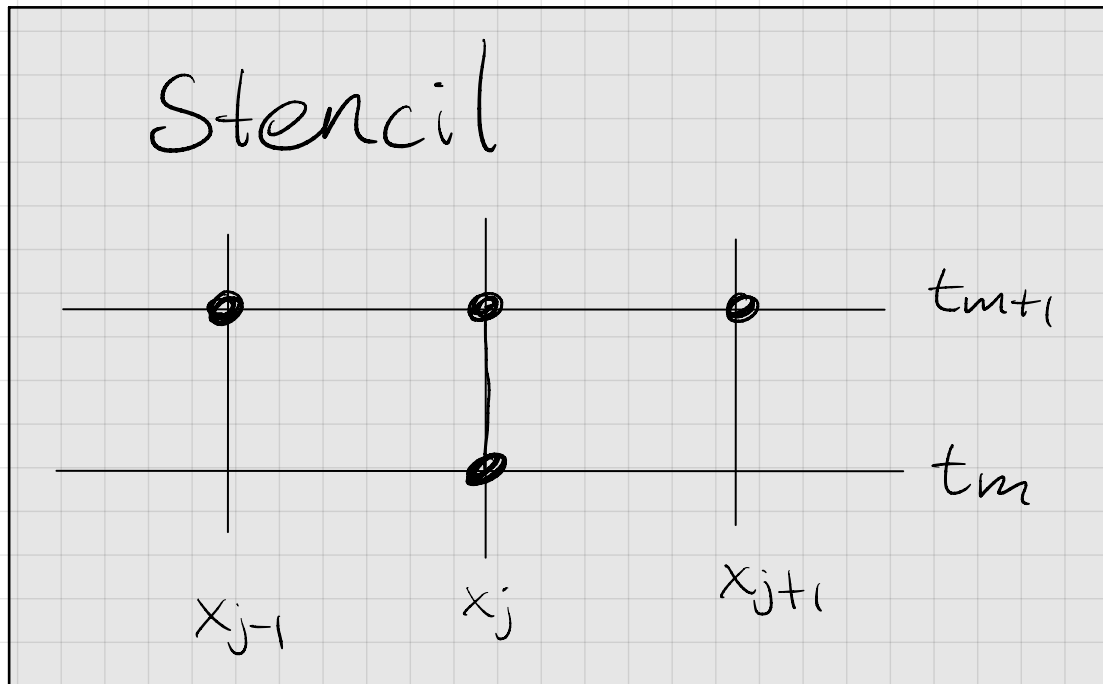
Scheme for $V_j^m \approx u(x_j, t_m)$:

$$\frac{V_j^{m+1} - V_j^m}{\Delta t} = \frac{V_{j-1}^{m+1} - 2V_j^{m+1} + V_{j+1}^{m+1}}{\Delta x^2} \quad (3)$$

for $j = 1, 2, \dots, n$ & $m \geq 0$,

Boundary cond: $V_0^m = V_{n+1}^m \quad \forall m \geq 0$

and $v_j^0 = f(x_j) \quad j = 1, 2, \dots, n$.



Thanks to $V^m = \begin{bmatrix} v_1^m \\ \vdots \\ v_n^m \end{bmatrix} \in \mathbb{R}^n$

and $A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \in \mathbb{R}^{n \times n}, \quad (4)$

the scheme (3) can be written

$$\left(I + \underbrace{\frac{\Delta t}{\Delta x^2}}_{=: r} A \right) V^{m+1} = V^m \quad (5)$$

For any $r \geq 0$, the matrix $(I + rA)$ is positive definite since $\forall x \in \mathbb{R}^n \setminus \{0\}$,

$$x^T (I + rA) x = \underbrace{x^T x}_{> 0} + \underbrace{r x^T A x}_{\geq 0} > 0.$$

$\Rightarrow (I + rA)$ is invertible $\Rightarrow$ (5) has unique solution.

Lemma

For any $r \geq 0$ the scheme (3) is Neumann stable.

Proof

Particular solutions of $u_t = u_{xx}$

are

$$u_k(x, t) = \underbrace{e^{-(k\pi)^2 t}}_{T_k(t)} e^{ik\pi x} \quad k \in \mathbb{Z}$$

and $\sup_{k \in \mathbb{Z}} |T_k(t)| = 1$

Inserting part solution

$$\bar{u}_{kj}^m = (a_k)^m e^{ik\pi x_j}$$

into (3):

$$\frac{a_k - 1}{\Delta t} (a_k)^m e^{ik\pi x_j} = \frac{e^{-ik\pi \Delta x} - 2 + e^{ik\pi \Delta x}}{\Delta x^2} (a_k)^{m+1} e^{ik\pi x_j}$$

we obtain

$$a_k = \frac{1}{1 + 4 \cdot r \sin^2\left(\frac{k\pi \Delta x}{2}\right)} \leq 1 \quad \forall k \in \mathbb{Z}$$

$$\Rightarrow \sup_{k \in \mathbb{Z}} |a_k|^m \leq 1 = \sup_{k \in \mathbb{Z}} |T_k(t_m)|$$

$$\forall m \geq 0$$

□

Observe: For heat eq.,

implicit scheme is stable for any

$\Delta t > 0$, while explicit scheme must satisfy

$$\frac{\Delta t}{\Delta x^2} < \frac{1}{2}.$$

$\Rightarrow$ Implicit scheme may be more efficient than explicit.

Discrete energy arguments

Have shown

$$E(t) = \int_0^1 (u(x,t))^2 dx \leq \int_0^1 (u(x,0))^2 dx = E(0)$$

for solution of heat eq. (2).

Consider solution of explicit scheme

$$\frac{V_j^{m+1} - V_j^m}{\Delta t} = \frac{V_{j-1}^m - 2V_j^m + V_{j+1}^m}{\Delta x^2}$$

$$\text{i.e. } V^{m+1} = (I - rA) V^m \quad (6)$$

with $r = \frac{\Delta t}{\Delta x^2}$ and A as in (4).

And energy

$$E_m := \Delta x \sum_{j=1}^n (V_j^m)^2 = \Delta x \underbrace{(V^m, V^m)}_{:= (V^m)^T V^m}$$

Then

$$E_{m+1} - E_m = \Delta x \left((V^{m+1}, V^{m+1}) - (V^m, V^m) \right) \quad (7)$$

Note that

$$\begin{aligned} (V^{m+1}, V^{m+1}) &\stackrel{(6)}{=} ((I-rA)V^m, (I-rA)V^m) \\ &= (V^m)^T (I-rA)^T (I-rA) V^m \\ &= (V^m)^T (I-rA)^2 V^m, \end{aligned}$$

$$\text{Since } A = A^T \Rightarrow (I-rA)^T = I-rA.$$

Eigenvalues of A are

$$\mu_k = 4 \sin^2\left(\frac{k\pi\Delta x}{2}\right) \quad k=1, \dots, n$$

So $\bar{\mu} = \max_k \mu_k \leq 4$ and

eigenvalues of $(I-rA)^2$ are

$$(1-r\mu_k)^2 \quad k=1, \dots, n.$$

If $r \leq \frac{1}{2}$, then $(1-r\mu_k)^2 \leq 1 \quad \forall k$ and

$$(V^m)^T (I - rA)^2 V^m \leq \max_k (1 - r\mu_k)^2 (V^m)^T V^m \\ \leq (V^m, V^m)$$

Inserting this into (7) yields

$$E_{m+1} \leq E_m \Rightarrow E_m \leq E_0 \quad \forall m \geq 0$$

Stability with respect to V^0 :

$$\Delta x \sum_{j=1}^n (V_j^m)^2 \leq \Delta x \sum_{j=1}^n (V_j^0)^2 \quad \forall m \geq 0$$

Implies uniqueness:

If $r \leq \frac{1}{2}$ and V^m and $\bar{V}^m$ are explicit scheme solutions of (1) with $V_j^0 = f(x_j)$ and $\bar{V}_j^m = \bar{f}(x_j)$, then

$$\Delta x \sum_{j=1}^n (V_j^m - \bar{V}_j^m)^2 \leq \Delta x \sum_{j=1}^n (f(x_j) - \bar{f}(x_j))^2 \quad \forall m \geq 0$$

Proof idea: Use that $W^m = V^m - \bar{V}^m$ is explicit scheme sol of (1) with $W^0 = V^0 - \bar{V}^0 = f - \bar{f}$, and apply energy argument on w .