

## Lecture 11

Wave eq. with Dirichlet BC cond:

$$\left. \begin{aligned} u_{tt} &= u_{xx} & x \in (0,1), t > 0 \\ u(0,t) &= u(1,t) = 0 & t \geq 0 \end{aligned} \right\} (1)$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x) \quad x \in (0,1) \quad (2)$$

### Particular solutions

Guess solution:  $u_k(x,t) = X_k(x) T_k(t)$   
with  $X_k \in C_0^2((0,1))$  &  $T_k \in C^2([0,\infty))$ .

Inserting into (1) yields

$$X_k(x) T_k''(t) = X_k''(x) T_k(t)$$

$$\Rightarrow \frac{T_k''(t)}{T_k(t)} = \frac{X_k''(x)}{X_k(x)} = -\lambda_k \quad (3)$$

for some  $\lambda_k \in \mathbb{R}$

Familiar eigenval problem

$$-X''(x) = \lambda_k X_k(x) \quad x \in (0,1)$$

$$X_k(0) = X_k(1) = 0$$

with eigenpairs

$$\left. \begin{aligned} \lambda_k &= (k\pi)^2 \\ X_k(x) &= \sin(k\pi x) \end{aligned} \right\} \quad k = 1, 2, \dots$$

Since  $\lambda_k$  is determined, (3) now gives ODE

$$T_k''(t) + (\pi k)^2 T_k = 0$$

General solution

$$T_k(t) = a_k \cos(k\pi t) + b_k \sin(k\pi t),$$

with  $a_k, b_k \in \mathbb{R}$ .

(Note for later:  $T_k(t)$  is linear combination of  $e^{ik\pi t}$  and  $e^{-ik\pi t}$ .)

Particular solutions of (1):

$$u_k(x,t) = (a_k \cos(k\pi t) + b_k \sin(k\pi t)) \sin(k\pi x)$$

for  $k=1, 2, \dots$

By linearity of PDE,

$$u(x, t) = \sum_{k=1}^N u_k(x, t)$$

$$= \sum_{k=1}^N (a_k \cos(k\pi t) + b_k \sin(k\pi t)) \sin(k\pi x)$$

also solves (1) with

$$u(x, 0) = \sum_{k=1}^N a_k \sin(k\pi x)$$

$$u_t(x, 0) = \sum_{k=1}^N k\pi b_k \sin(k\pi x)$$

$$u(x, 0) = f(x)$$

$$\Rightarrow a_k = 2 \langle f, \sin(k\pi x) \rangle = 2 \int_0^1 f(x) \sin(k\pi x) dx$$

$$\text{and } u_t(x, 0) = g(x)$$

$$\Rightarrow b_k = \frac{2}{k\pi} \int_0^1 g(x) \sin(k\pi x) dx$$

} (4)

For sine series initial data, we can solve (1) - (2):

Ex

$$\text{Let } f(x) = 5 \sin(12\pi t) + 3 \sin(42\pi x)$$

$$\text{and } g(x) = 17 \sin(3\pi x).$$

Solution

$$\begin{aligned} u(x,t) = & 5 \cos(12\pi t) \sin(12\pi x) \\ & + 3 \cos(42\pi t) \sin(42\pi x) \\ & + \frac{17}{3\pi} \sin(3\pi t) \sin(3\pi x). \end{aligned}$$

Formal solution when  $N \rightarrow \infty$

$$u(x,t) = \sum_{k=1}^{\infty} (a_k \cos(k\pi t) + b_k \sin(k\pi t)) \sin(k\pi x)$$

with  $a_k, b_k$  determined by (4).

## Energy argument

Let  $u$  be sol of (1)-(2) and assume that  $u \in C^2([0,1] \times [0,\infty))$

(= twice cont. differentiable functions).

Consider

$$E(t) := \int_0^1 u_x^2(x,t) + u_t^2(x,t) dx.$$

Note that  $u_x^2, u_t^2 \in C^{1,1}([0,1] \times [0,\infty))$ ,  
so that

$$E'(t) = \frac{d}{dt} \int_0^1 u_x^2 + u_t^2 dx$$

Lecture 8

$$\stackrel{\downarrow}{=} 2 \int_0^1 u_x u_{xt} + u_{tt} u_t dx \quad (5)$$

Since  $u \in C^2$ , it holds that

$$u_{xt} = \frac{\partial}{\partial t} \left( \frac{\partial}{\partial x} u \right) = \frac{\partial}{\partial x} \left( \frac{\partial}{\partial t} u \right) = u_{tx},$$

which implies

$$\begin{aligned}
\int_0^1 u_x u_{xt} dx &= \int_0^1 u_x u_{tx} dx \\
&= \int_0^1 \frac{\partial}{\partial x} (u_x u_t) dx - \int_0^1 u_{xx} u_t \\
&= \underbrace{\left[ u_x(x,t) u_t(x,t) \right]_0^1}_{=0} - \int_0^1 u_{tt} u_t dt
\end{aligned}$$

Since  $u(0,t) = 0 \quad \forall t \geq 0 \Rightarrow u_t(0,t) = 0$   
 & similarly  $u_t(1,t) = 0$ .

Inserting into (5) yields

$$\begin{aligned}
E'(t) &= 2 \int_0^1 u_x u_{xt} + u_t u_{tt} dx \\
&= 0
\end{aligned}$$

$$\Rightarrow E(t) = E(0) \quad \text{for } t \geq 0. \quad (6)$$

## Stability

Let  $u$  and  $\tilde{u}$  be  $C^2$  solutions of (1) with

|                   |                                 |
|-------------------|---------------------------------|
| $u(x,0) = f(x)$   | $\tilde{u}(x,0) = \hat{f}(x)$   |
| $u_t(x,0) = g(x)$ | $\tilde{u}_t(x,0) = \hat{g}(x)$ |

Then

$w = u - \tilde{u}$  solves (1) with

$$w(x, 0) = (f - \tilde{f})(x)$$

$$w_t(x, 0) = (g - \tilde{g})(x)$$

and (6)  $\Rightarrow$

$$\begin{aligned} & \int_0^1 ((u - \tilde{u})_x(x, t))^2 + ((u - \tilde{u})_t(x, t))^2 dx \\ &= \int_0^1 ((f - \tilde{f})_x(x))^2 + ((g - \tilde{g})(x))^2 dx \quad (7) \end{aligned}$$

Uniqueness  $\rightarrow$

If above  $u, \tilde{u} \in C^2$  both solve  
(1) - (2) (so  $f = \tilde{f}$  and  $g = \tilde{g}$ )

then  $u = \tilde{u}$ .

Proof By (7)

$$\int_0^1 ((u - \tilde{u})_x(x, t))^2 + ((u - \tilde{u})_t(x, t))^2 dx = 0$$

for all  $t \geq 0$ .

$$\Rightarrow u_x(x,t) = \tilde{u}_x(x,t) \quad \forall x,t$$

$$\begin{aligned} \Rightarrow u(x,t) &= \underbrace{u(0,t)}_{=0} + \int_0^x u_x(\gamma,t) d\gamma \\ &= \int_0^x \tilde{u}_x(\gamma,t) d\gamma = \tilde{u}(x,t) \quad \forall x,t \end{aligned}$$

□

Numerical method for

$$u_{tt} = u_{xx} \quad x \in (0,1), t > 0$$

$$u(0,t) = u(1,t) = 0 \quad t \geq 0$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

Approximations:

$$u_{tt}(x,t) \approx \frac{u(x,t-\Delta t) - 2u(x,t) + u(x,t+\Delta t)}{\Delta t^2}$$

$$u_{xx}(x,t) \approx \frac{u(x-\Delta x,t) - 2u(x,t) + u(x+\Delta x,t)}{\Delta x^2}$$

$$\text{Num solution } v_j^m \approx u(x_j, t_m)$$

$$\text{with } \Delta x = \frac{1}{n+1}, \quad x_j = j\Delta x \quad j=0,1,\dots,n+1$$

$$\text{and } t_m = m\Delta t \quad \text{for some } \Delta t > 0.$$

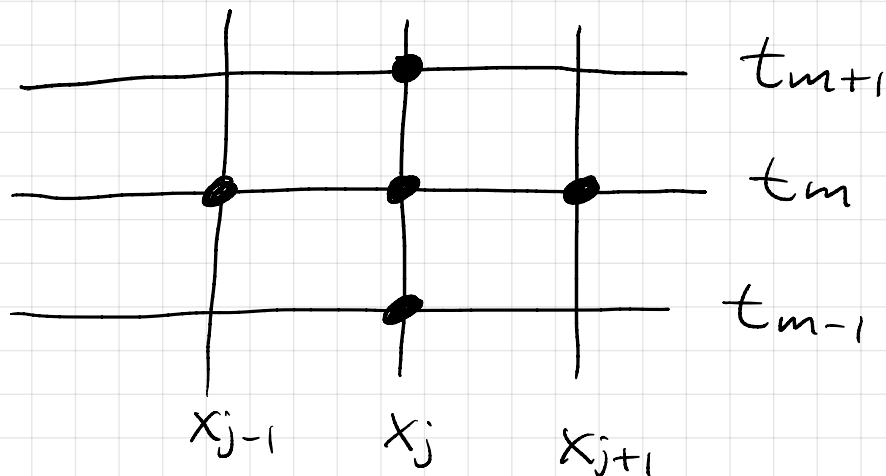
Gives scheme

$$\frac{V_j^{m-1} - 2V_j^m + V_j^{m+1}}{\Delta t^2} = \frac{V_{j-1}^m - 2V_j^m + V_{j+1}^m}{\Delta x^2} \quad (8)$$

for  $j = 1, \dots, n$  and  $m \geq 1$ .

Boundary cond:  $V_0^m = V_{n+1}^m = 0 \quad m \geq 0$ .

Stencil



Initial conditions needed for  $v^0$  &  $v'$ :

$$v_j^0 = f(x_j) \quad j = 1, \dots, n$$

How to approximate  $u_t(x, 0) = g(x)$ ?

$$u(x, \Delta t) = u(x, 0) + \Delta t \cdot u_t(x, 0) + \frac{\Delta t^2}{2} u_{tt}(x, 0) + O(\Delta t^3)$$

$$\approx f(x) + \Delta t g(x) + \frac{\Delta t^2}{2} u_{xx}(x, 0)$$

We use

$$V_j^1 = f(x_j) + \Delta t g(x_j) + \frac{\Delta t^2}{2\Delta x^2} (V_{j-1}^0 - 2V_j^0 + V_{j+1}^0)$$
$$j=1, \dots, n.$$

With  $V^m = \begin{bmatrix} V_1^m \\ \vdots \\ V_n^m \end{bmatrix}$ ,  $A = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \\ & & -1 & 2 \end{bmatrix} \in \mathbb{R}^{n \times n}$

and  $r = \frac{\Delta t}{\Delta x}$ , the scheme (8) becomes

$$V^{m+1} = (2I - r^2 A) V^m - V^{m-1} \quad m \geq 1 \quad (9)$$

Neumann stability

$$u_k(x, t) = \sin(k\pi x) e^{\pm i k \pi t} \quad k \in \mathbb{N}$$

are particular solutions of (1).

For (9),

$$\bar{u}_k^m = (a_k)^m V_k \in \mathbb{R}^n \quad \text{for } k=1, \dots, n$$

with  $V_k = (\sin(k\pi x_1), \dots, \sin(k\pi x_n))^T$   
are part. solutions.

To determine  $a_k$ , recall that

$$A v_k = 4 \sin^2\left(\frac{k\pi\Delta x}{2}\right) v_k$$

Inserting  $\bar{u}_k^m$  into (9) yields

$$\bar{u}_k^{m+1} = (2I - r^2 A) \bar{u}_k^m - \bar{u}_k^{m-1}$$

$$(a_k)^{m+1} v_k = \underbrace{\left(2 - 4r^2 \sin^2\left(\frac{k\pi\Delta x}{2}\right)\right)}_{=: s} (a_k)^m - (a_k)^{m-1} v_k$$

Eq for  $a_k$ :

$$a_k^2 - (2 - s) a_k + 1 = 0$$

$$\Rightarrow a_k = \frac{2 - s \pm \sqrt{s(s-4)}}{2}$$

$\Rightarrow$  Both roots  $|a_{k\pm}| \leq 1$  iff

$$s \leq 4$$

And  $r \leq 1 \Rightarrow s \leq 4$ .

Conclusion: If  $\frac{\Delta t}{\Delta x} \leq 1$  (CFL)  
then

$$\max_{k=1, \dots, n} |a_{k\pm}|^m \leq 1 = \sup_{k \in \mathbb{N}} |e^{\pm i k \pi t_m}| \quad m \geq 0$$

and the scheme is Neumann  
stable.

