

Lecture 12

For

$$\left. \begin{aligned} u''(x) + a(x)u'(x) &= 0 & x \in (0,1) \\ u(0) &= u_0 \text{ \& } u(1) = u_1 \end{aligned} \right\} (1)$$

with $a \in C([0,1])$, we will show that

$$\min(u_0, u_1) \leq u(x) \leq \max(u_0, u_1) \quad \forall x \in [0,1] \quad (2)$$

(2) is called a maximum principle.

Recall that if $x_0 \in (0,1)$ is a local max of $u(x)$, then

$$u'(x_0) = 0 \quad \& \quad u''(x_0) \leq 0. \quad (3)$$

Helpful result:

Lemma Let

$v \in C^2((0,1)) \cap C([0,1])$ satisfy

$$v''(x) + a(x)v'(x) > 0 \quad x \in (0,1) \quad (4)$$

Then

$$v(x) \leq \max_x(v(0), v(1)) \quad \forall x \in [0,1].$$

Proof Assume $x_0 \in (0,1)$ is a local max of v . Then (3) $\Rightarrow$

$$V'(x_0) = 0 \quad \& \quad V''(x_0) \leq 0,$$

which implies that

$$V''(x_0) + a(x) V'(x_0) \leq 0,$$

contradicting (4). So v has no maximizer in $(0,1)$. $\square$

Suppose next that $u \in C^2((0,1)) \cap C([0,1])$ solves (1) and consider

$$V_\epsilon(x) := u(x) + \epsilon e^{(1+c)x} \quad \text{for } \epsilon > 0,$$

$$\text{where } c := \max_{x \in [0,1]} |a(x)|.$$

Then

$$V_\epsilon''(x) + a(x) V_\epsilon'(x)$$

$$= \underbrace{u''(x) + a(x) u'(x)}_{=0} + \underbrace{\epsilon(1+c)(1+c+a(x))}_{>0} e^{(1+c)x}$$

$$> 0$$

Previous lemma implies that

$$\begin{aligned}V_\epsilon(x) &\leq \max(V_\epsilon(0), V_\epsilon(1)) \\&= \max(u_0 + \epsilon, u_1 + \epsilon e^{1+c}) \\&\leq \max(u_0, u_1) + \epsilon e^{1+c}.\end{aligned}$$

Furthermore

$$u(x) \leq V_\epsilon(x) \leq \max(u_0, u_1) + \epsilon e^{1+c}.$$

Limit $\epsilon \rightarrow 0$ yields

$$u(x) \leq \max(u_0, u_1).$$

What about lower bound

$$u(x) \geq \min(u_0, u_1) \quad ?$$

Consider $w(x) = -u(x)$.

$$\text{Then } w''(x) + a(x)w(x) = 0$$

$$w(0) = -u_0 \text{ \& } w(1) = -u_1$$

Above argument gives

$$w(x) \leq \max(w(0), w(1))$$

$$\Rightarrow -u(x) \leq \max(-u_0, -u_1) \\ = -\min(u_0, u_1)$$

$$\Rightarrow u(x) \geq \min(u_0, u_1).$$

Stability and uniqueness

Assume that $u, \tilde{u} \in C^2((0,1)) \cap C([0,1])$.
both solve $u'' + a(x)u' = 0$ with
 $u(0) = u_0, u(1) = u_1$ & $\tilde{u}(0) = \tilde{u}_0, \tilde{u}(1) = \tilde{u}_1$

Then

$e := u - \tilde{u}$ solves

$$e''(x) + a(x)e'(x) = 0$$

and maximum principle $\Rightarrow$

$$\min(\overbrace{u_0 - \tilde{u}_0}^{e(0)}, \overbrace{u_1 - \tilde{u}_1}^{e(1)}) \leq e(x) \leq \max(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1)$$

$$\Rightarrow |u(x) - \tilde{u}(x)| \leq \max(|u_0 - \tilde{u}_0|, |u_1 - \tilde{u}_1|)$$

$\Rightarrow$ (1) has at most one solution in
 $C^2((0,1)) \cap C([0,1])$.

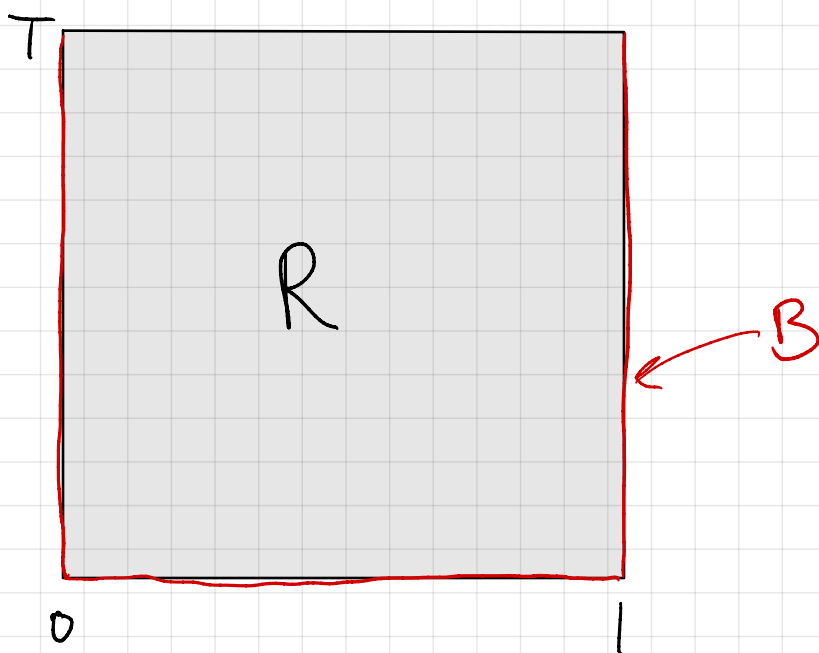
Linear heat eq

$$\left. \begin{aligned} u_t &= u_{xx} & x \in (0,1), t \in (0,T] \\ u(0,t) &= u_l(t) & u(1,t) &= u_r(t) & t \in [0,T] \\ u(x,0) &= f(x) & x \in (0,1) \end{aligned} \right\} (5)$$

Domain of u : $R = [0,1] \times [0,T]$

"Lower" boundary of R :

$$B = R \setminus (0,1) \times (0,T].$$



Theorem

Suppose $u \in C(R) \cap C^{2,1}([0,1] \times (0,T])$ is a solution of (5). Then

$$\inf_{(x,t) \in B} u(x,t) \leq u(x,t) \leq \sup_{(x,t) \in B} u(x,t) \quad \forall (x,t) \in B.$$

Equivalently,

$$\inf_{(x,t) \in B} (f(x), u_\ell(t), u_r(t)) \leq u(x,t) \leq \sup_{(x,t) \in B} (f(x), u_\ell(t), u_r(t)). \quad (6)$$

Proof Main steps

① Verify (6) for $v(x,t)$ s.t.

$$V_t < V_{xx}$$

② Prove for $V^\epsilon(x,t) := u(x,t) + \epsilon x^2$ and take limit $\epsilon \rightarrow 0$.

① Suppose $v \in C^{2,1}([0,1] \times (0,T]) \cap C(R)$

satisfies $V_t < V_{xx} \quad x \in (0,1), t \in (0,T]$

If $(x_0, t_0) \in (0,1) \times (0,T)$ is a local max of v , then $V_t(x_0, t_0) = 0$ and

$V_{xx}(x_0, t_0) \leq 0$. Leads to contradiction

$$V_t(x_0, t_0) \geq V_{xx}(x_0, t_0).$$

If (x_0, T) with $x_0 \in (0, 1)$ is a local max,
then $V_t(x_0, T) \geq 0$ & $V_{xx}(x_0, T) \leq 0$.

Contradiction: $V_t(x_0, T) \geq V_{xx}$

Conclusion: V has no interior-point
maximizer $(x_0, t_0) \in R \setminus B$, so

$$V(x, t) \leq \sup_{(x, t) \in B} V(x, t)$$

② Consider $V^\epsilon(x, t) := u(x, t) + \epsilon x^2$
for $\epsilon > 0$.

Then

$$V_t^\epsilon - V_{xx}^\epsilon = \underbrace{u_t - u_{xx}}_{=0} - 2\epsilon < 0$$

Step ① implies that

$$V^\epsilon(x, t) \leq \sup_{(x, t) \in B} V^\epsilon(x, t)$$

Since $u \leq v^\epsilon = u + \epsilon x^2 \leq u + \epsilon$,

$$u(x,t) \leq v^\epsilon(x,t) \leq \sup_{(x,t) \in B} v^\epsilon(x,t)$$

$$\leq \sup_{(x,t) \in B} u(x,t) + \epsilon$$

Limit $\epsilon \rightarrow 0$ yields

$$u(x,t) \leq \sup_{(x,t) \in B} u(x,t) \quad \forall (x,t) \in R.$$

Lower bound in (6):

The function $w(x,t) := -u(x,t)$

solves $w_t = w_{xx}$ and by ①-②,

$$w(x,t) \leq \sup_{(x,t) \in B} w(x,t)$$

$$= \sup_{(x,t) \in B} -u(x,t)$$

$$= -\inf_B u$$

$$\Rightarrow u(x,t) \geq \inf_B u \quad \forall (x,t) \in R$$



Uniqueness and stability

If $u, \tilde{u} \in C^{2,1}([0,1] \times (0,T]) \cap C(\bar{Q})$

solve $u_t = u_{xx}$ with

$$u(0,t) = u_\ell(t)$$

$$u(1,t) = u_r(t)$$

$$u(x,0) = f(x)$$

and

$$\tilde{u}(0,t) = \tilde{u}_\ell(t)$$

$$\tilde{u}(1,t) = \tilde{u}_r(t)$$

$$\tilde{u}(x,0) = \tilde{f}(x)$$

Then $e := u - \tilde{u}$ solves

$$e_t = e_{xx} \quad \text{and} \quad (6) \Rightarrow$$

$$|e(x,t)| \leq \sup_{(x,t) \in B} |e(x,t)|$$

$$= \sup_{(x,t) \in B} (|f(x) - \tilde{f}(x)|, |u_\ell(t) - \tilde{u}_\ell(t)|, |u_r(t) - \tilde{u}_r(t)|)$$

Conclusion: The PDE (5) has at most one solution in $C^{2,1}([0,1] \times (0,T]) \cap C(\bar{Q})$.

Discrete maximum principles

Schemes for (5):

$$\frac{V_j^{m+1} - V_j^m}{\Delta t} = \frac{V_{j-1}^m - 2V_j^m + V_{j+1}^m}{\Delta x^2} \quad (\text{Explicit})$$

$$\frac{V_j^{m+1} - V_j^m}{\Delta t} = \frac{V_{j-1}^{m+1} - 2V_j^{m+1} + V_{j+1}^{m+1}}{\Delta x^2} \quad (\text{Implicit})$$

both for $j=1, \dots, n$, $m \geq 0$.

Boundary cond for both schemes

$$V_0^m = u_l(t_m), \quad V_{n+1}^m = u_r(t_m) \quad m \geq 0$$

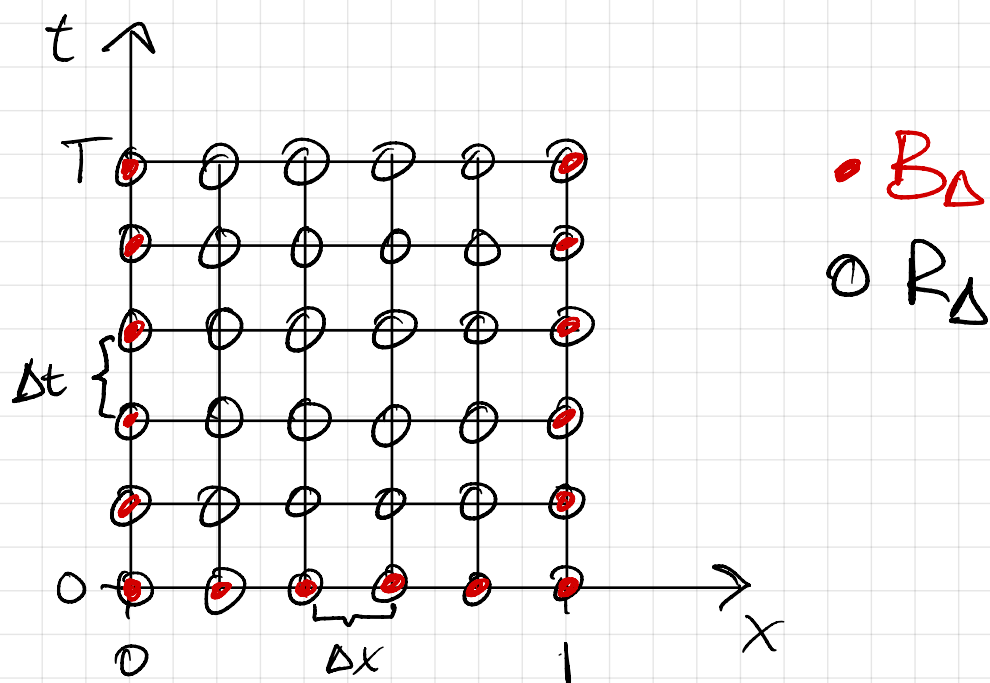
$$V_j^0 = f(x_j)$$

Discrete geometry:

Domain $R_\Delta = \{(x_j, t_m) \mid x_j \in [0,1] \text{ and } t_m \in [0,T]\}$

"Lower" boundary

$$B_\Delta = \{(x_j, t_m) \in R_\Delta \mid t_m = 0\} \cup \{(x_j, t_m) \in R_\Delta \mid x_j \in \{0,1\}\}$$



Let

$$V^- = \min_{(x_j, t_m) \in B_\Delta} (f(x_j), u_\ell(t_m), u_r(t_m))$$

$$V^+ = \max_{(x_j, t_m) \in B_\Delta} (f(x_j), u_\ell(t_m), u_r(t_m))$$

and $r = \frac{\Delta t}{\Delta x^2}$.

Theorem If $r \leq \frac{1}{2}$, then the explicit scheme satisfies

$$V^- \leq V_j^m \leq V^+ \quad \forall (x_j, t_m) \in R_\Delta.$$

And for any $\Delta x = \frac{1}{n+1}$ and $\Delta t > 0$ (also when $r > \frac{1}{2}$), it holds for the implicit scheme that

$$V^- \leq V_j^m \leq V^+ \quad \forall (x_j, t_m) \in R_\Delta$$

Proof upper bound

Explicit scheme: Write scheme as

$$V_j^{m+1} = r V_{j-1}^m + (1-2r) V_j^m + r V_{j+1}^m \quad (7)$$

Induction: Assume that $\max_{j=0,1,\dots,n+1} V_j^m \leq V^+$ holds for some $m \geq 0$. Then (7) $\Rightarrow$

$$V_j^{m+1} \leq r V^+ + (1-2r) V^+ + r V^+ = V^+$$

for all $j = 1, \dots, n$.

$$\text{And } V_0^{m+1} = V(x_0, t_{m+1}) \leq V^+ \\ V_{n+1}^{m+1} = V(x_{n+1}, t_{m+1}) \leq V^+$$

hold since $(x_0, t_{m+1}), (x_{n+1}, t_{m+1}) \in B_\Delta$.
 $\Rightarrow \max_{j=0,1,\dots,n+1} V_j^{m+1} \leq V^+$

Upper bound in theorem holds by induction.

Implicit scheme: Write scheme as
 $(1+2r) V_j^{m+1} = V_j^m + r(V_{j-1}^{m+1} + V_{j+1}^{m+1}) \quad (8)$

Assume $\max_{j=0,1,\dots,n+1} V_j^m \leq V^+$ holds for
some $m \geq 0$. Then (8) $\Rightarrow$

$$(1+2r) V_j^{m+1} \leq V^+ + 2r \max_{j=0,1,\dots,n+1} V_j^{m+1} \quad (9)$$

holds for all $j=1,2,\dots,n$. Since

$V_0^{m+1} \leq V^+$ and $V_{n+1}^{m+1} \leq V^+$ (9) also

holds for $j=0$ and $j=n+1$, so that

$$\cancel{(1+2r)} \max_{j=0,1,\dots,n+1} V_j^{m+1} \leq V^+ + \cancel{2r} \max_{j=0,1,\dots,n+1} V_j^{m+1}$$

$$\Rightarrow \max_{j=0,1,\dots,n+1} V_j^{m+1} \leq V^+.$$

Upper bound in theorem holds by induction.

Lower bounds can be proved similarly 

Remark The theorem gives an alternative stability estimate to von Neumann's.