

Lecture 13

Nonlinear heat eq.

$$\left. \begin{aligned} u_t &= (K(u)u_x)_x & x \in (0,1), \quad t \in (0,T] \\ u(0,t) &= u_l(t) \quad u(1,t) = u_r(t) & t \in [0,T] \\ u(x,0) &= f(x) & x \in (0,1) \end{aligned} \right\} (1)$$

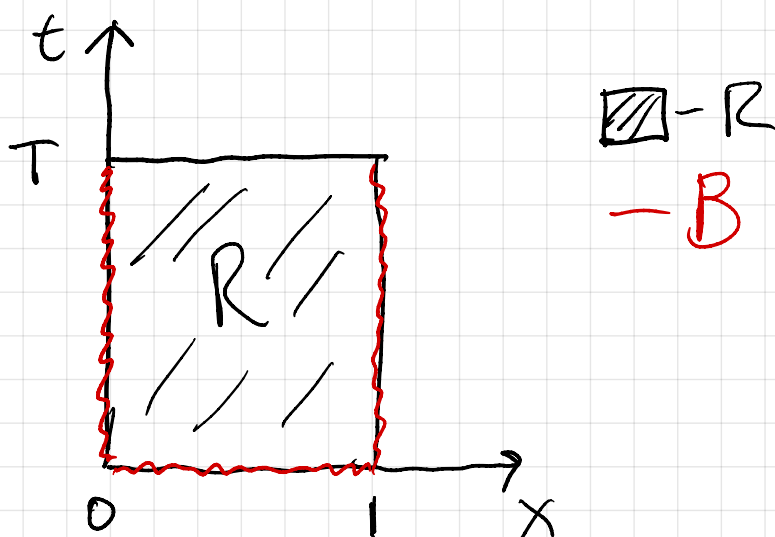
Where we assume $K \in C^1(\mathbb{R})$ and

$$0 < K_0 \leq K(x) \leq K_0 \quad \forall x \in \mathbb{R} \quad (2).$$

Let $R = [0,1] \times [0,T]$ and

"lower" boundary

$$B = R \setminus (0,1) \times (0,T]$$



Theorem If

$u \in C^{2,1}((0,1) \times (0,T]) \cap C(\bar{R})$ solves (1), then

$$\inf_{(x,t) \in B} (f(x), u_l(t), u_r(t)) \leq u(x,t) \leq \sup_{(x,t) \in B} (f(x), u_l(t), u_r(t))$$

for all $(x,t) \in R$.

Proof

For any $\epsilon > 0$, let $V^\epsilon(x,t) := u(x,t) - \epsilon t$.

Then

$$V_t^\epsilon = u_t - \epsilon$$

$$= K(u) u_{xx} + k'(u) (u_x)^2 - \epsilon$$

$$= K(V^\epsilon + \epsilon t) V_{xx}^\epsilon + k'(V^\epsilon + \epsilon t) (V_x^\epsilon)^2 - \epsilon$$

$$\Rightarrow V_t^\epsilon < K(V^\epsilon + \epsilon t) V_{xx}^\epsilon + k'(V^\epsilon + \epsilon t) (V_x^\epsilon)^2 \quad (3)$$

Suppose $(x_0, t_0) \in (0,1) \times (0,T] = R \setminus B$ is a local maximizer of V^ϵ . Then

$$V_x^\epsilon(x_0, t_0) = 0 \text{ \& \> } V_{xx}^\epsilon(x_0, t_0) \leq 0$$

$$\text{and } V_t^\epsilon(x_0, t_0) = \begin{cases} \geq 0 & \text{if } t_0 = T \\ 0 & \text{if } t_0 \in (0, T) \end{cases}$$

$\Rightarrow$

$$\underbrace{V_t^\epsilon(x_0, t_0)}_{\geq 0} \geq \underbrace{\overbrace{K(V^\epsilon(x_0, t_0) + \epsilon t)}^{(2) > 0}}_{\leq 0} \underbrace{V_{xx}^\epsilon(x_0, t_0)}_{\leq 0} + \underbrace{K'(V^\epsilon(x_0, t_0) (V_x^\epsilon(x_0, t_0))^2)}_{= 0}$$

which contradicts (3). So V^ϵ has no maximizer in the interior $R \setminus B$ and

$$V^\epsilon(x, t) \leq \sup_{(x, t) \in B} V^\epsilon(x, t) \quad \forall (x, t) \in R$$

Recalling $V^\epsilon = u - \epsilon t \leq u$ yields

$$u(x, t) - \epsilon t \leq \sup_{(x, t) \in B} u(x, t)$$

$$\Rightarrow u(x, t) \leq \sup_{(x, t) \in B} u(x, t) + \epsilon T$$

Limit $\epsilon \rightarrow 0$ gives upper bound.

For lower bound, consider

$$W(x, t) = -u(x, t).$$



Note: This maximum principle cannot be used to derive stability and uniqueness of solutions the way done for the linear heat equation, since principle of superposition does not apply to nonlinear PDE.

Numerical method for (1):

$V_j^m = U(x_j, t_m)$ with mesh (x_j, t_m) as before.

Explicit scheme (Exercise 4.20)

$$V_j^{m+1} = r k_{j-1/2}^m V_{j-1}^m + (1 - r(k_{j-1/2}^m + k_{j+1/2}^m)) V_j^m + r k_{j+1/2}^m V_{j+1}^m$$

for $j=1, \dots, n$, $m \geq 0$, with $r = \frac{\Delta t}{\Delta x^2}$,

$$k_{j+1/2}^m = \frac{k(V_j^m) + k(V_{j+1}^m)}{2}$$

Boundary conditions

$$\begin{aligned} V_0^m &= U_e(t_m) & V_{n+1}^m &= U_r(t_m) & m &\geq 0 \\ V_j^0 &= f(x_j) & j &= 1, \dots, n. \end{aligned}$$

See lecture 12 for domain R_Δ and lower boundary B_Δ .

Let

$$V^+ = \max_{(x_j, t_k) \in B_\Delta} (f(x_j), u_\ell(t_k), u_r(t_k))$$

$$V^- = \min_{(x_j, t_k) \in B_\Delta} (f(x_j), u_\ell(t_k), u_r(t_k)).$$

Theorem If $\frac{\Delta t}{\Delta x^2} K_0 \leq \frac{1}{2}$ holds
(K_0 from (2)), then

$$V^- \leq V_j^m \leq V^+ \quad \forall (x_j, t_m) \in R_\Delta$$

Proof Assume $\max_{j=0,1,\dots,n+1} V_j^m \leq V^+$

holds for some $m \geq 0$. Then,

since $k_{j+1/2}^m \leq K_0 \quad \forall j$ and

$$1 - r(k_{j-1/2}^m + k_{j+1/2}^m) \geq 1 - 2rK_0 \geq 0,$$

we obtain

$$V_j^{m+1} = r k_{j-1/2}^m V_{j-1}^m + (1 - r(k_{j-1/2}^m + k_{j+1/2}^m)) V_j^m + r k_{j+1/2}^m V_{j+1}^m$$

$$\leq r K_0 V^+ + (1 - 2r K_0) V^+ + r K_0 V^+$$

$$= V^+$$

for all $j=1, \dots, n$,

and it also holds for $j=0$ and $j=n+1$ since $(x_0, t_{m+1}), (x_{n+1}, t_{m+1}) \in B_\Delta$.

Hence

$$\max_{j=0,1,\dots,n+1} V_{j+1}^{m+1} \leq V^+,$$

and upper bound follows by induction.

Lower bound can be proved similarly

□