

Lecture 14

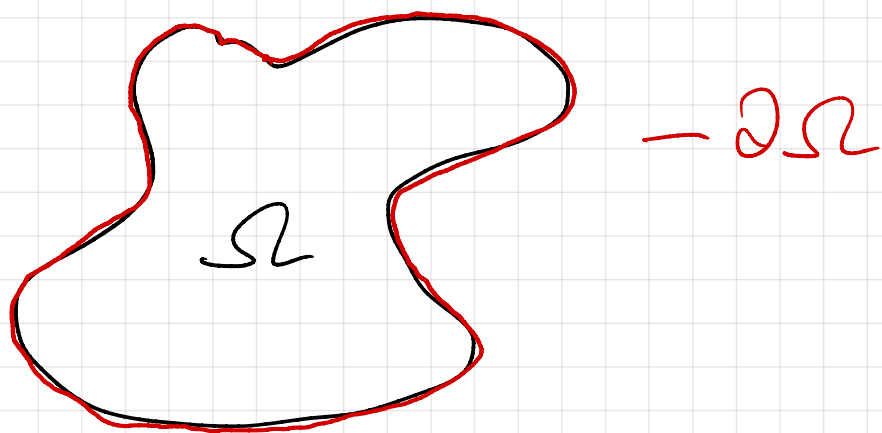
The d -dimensional Laplace operator Δ is defined by

$$\Delta u(x) := \sum_{j=1}^d u_{x_j x_j}(x). \quad (x \in \mathbb{R}^d)$$

Let $\Omega \subset \mathbb{R}^2$ be an open, connected and bounded domain.

Closure $\bar{\Omega}$

Boundary $\partial\Omega = \bar{\Omega} \setminus \Omega$.



A function $u \in C^2(\Omega) \cup C(\bar{\Omega})$ is harmonic in Ω if

$$\Delta u(x, y) = 0 \quad \forall (x, y) \in \Omega$$

Ex Harmonic functions

1d: All

$$u(x) = c_0 + c_1 x \quad , \quad c_k \in \mathbb{R}$$

2d: All functions

$$u(x, y) = c_0 + c_1 x + c_2 y + c_3 xy + c_4 (x^2 - y^2) \quad c_k \in \mathbb{R}$$

but also many more, like

$$e^x \sin(y) \quad \text{and} \quad \log(x^2 + y^2)$$

Maximum principles

In 1d, the ODE

$$u''(x) = 0 \quad \text{in } (0, 1)$$

can be written

$$u''(x) + a(x)u'(x) = 0$$

with $a \equiv 0$. Know from Lecture 12

that

$$\min(u(0), u(1)) \leq u(x) \leq \max(u(0), u(1)) \quad \forall x \in [0, 1]$$

Extension to 2d:

Theorem 1 If u is harmonic in Ω ,
then $\forall (x,y) \in \Omega$,

$$\min_{\partial\Omega} u \leq u(x,y) \leq \max_{\partial\Omega} u$$

Proof For any $\epsilon > 0$, let

$$v^\epsilon(x,y) := u(x,y) + \epsilon(x^2 + y^2)$$

$$\Delta u = 0$$

$$\Rightarrow \Delta v^\epsilon(x,y) = 4\epsilon > 0 \quad \forall (x,y) \in \Omega. \quad (1)$$

Suppose interior point $(x_0, y_0) \in \Omega$
is local max of v^ϵ . Then

$$v_{xx}^\epsilon(x_0, y_0) \leq 0 \text{ \& \& } v_{yy}^\epsilon(x_0, y_0) \leq 0.$$

$$\Rightarrow \Delta v^\epsilon(x_0, y_0) \leq 0.$$

This contradicts (1), so v^ϵ has no
maximizer in Ω and

$$v^\epsilon \leq \max_{\partial\Omega} v^\epsilon.$$

$$\text{Let } C_\Omega := \max_{(x,y) \in \Omega} x^2 + y^2 < \infty$$

($C_\Omega < \infty$ since Ω bounded) and note that

$$u(x,y) \leq \underbrace{V^\epsilon(x,y)}_{= u(x,y) + \epsilon(x^2+y^2)}$$

$$\leq \max_{(x,y) \in \partial\Omega} V^\epsilon(x,y)$$

$$\leq \max_{(x,y) \in \partial\Omega} u(x,y) + \epsilon C_\Omega$$

Taking $\epsilon \rightarrow 0$ gives upper bound.
Lower bound proved similarly.

□

Note: key step of proof is to find V^ϵ s.t. $\Delta V^\epsilon > 0$ & $V^\epsilon \rightarrow u$. This is possible under the weaker condition

$$\Delta u \geq 0 \quad \text{in } \Omega \quad (\text{subharmonic in } \Omega)$$

Corollary 1

If u is subharmonic, then

$$u(x,y) \leq \max_{\partial\Omega} u \quad \forall (x,y) \in \Omega$$

Application of Thm 1 (uniqueness)

Consider Poisson eq.

$$\left. \begin{aligned} -\Delta u &= f & \text{in } \Omega \\ u &= g & \text{on } \partial\Omega \end{aligned} \right\} (2)$$

Assume that $u, \tilde{u} \in C^2(\Omega) \cap C(\bar{\Omega})$ are solutions of (2) with the same f and g functions. Then

$w := u - \tilde{u}$ solves the PDE

$$\Delta w = \Delta u - \Delta \tilde{u} = 0 \quad \text{in } \Omega$$

$$w = g - g = 0 \quad \text{on } \partial\Omega.$$

Since w is harmonic, Thm 1 implies that

$$0 = \min_{\partial\Omega} w \leq w(x, y) \leq \max_{\partial\Omega} w = 0$$

$$\Rightarrow w = 0 \Rightarrow u = \tilde{u}$$

Discrete harmonic functions

Continuous domain

$$\Omega = (0,1) \times (0,1), \quad \overline{\Omega} = [0,1] \times [0,1]$$

with $\partial\Omega = \overline{\Omega} \setminus \Omega$.

(Goal: numerical method for solving (2).)

For $h = \frac{1}{n+1}$ and points

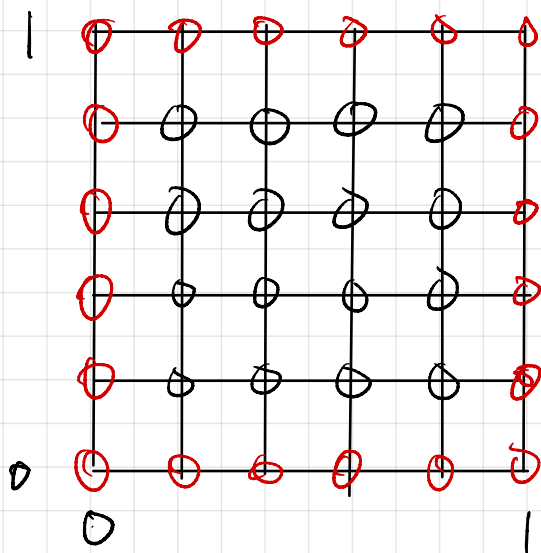
$$(x_j, y_k) = (jh, kh), \quad 0 \leq j, k \leq n+1,$$

the discrete domain / grid is given by

$$\Omega_h = \{(x_j, y_k) \mid 1 \leq j, k \leq n\} \quad \text{and}$$

$$\overline{\Omega}_h = \{(x_j, y_k) \mid 0 \leq j, k \leq n+1\},$$

with $\partial\overline{\Omega}_h = \overline{\Omega}_h \setminus \Omega_h$.



○ - Ω_h

○ - $\partial\overline{\Omega}_h$

A grid function $V: \bar{\Omega}_h \rightarrow \mathbb{R}$
 (or $V: \bar{\Omega}_h \rightarrow \mathbb{R}^{(n+2)^2}$ for full vector of values)
 we write $V_{j,k} := V(x_j, y_k)$.

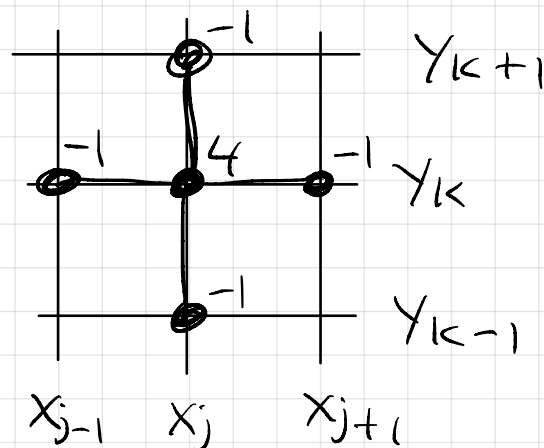
The Discrete "minus Laplacean":

$$\begin{aligned} (L_h V)(x_j, y_k) &= \frac{1}{h^2} \left[(-V_{j-1,k} + 2V_{j,k} - V_{j+1,k}) + (-V_{j,k-1} + 2V_{j,k} - V_{j,k+1}) \right] \\ &= \frac{1}{h^2} \left[4V_{j,k} - (V_{j-1,k} + V_{j+1,k} + V_{j,k-1} + V_{j,k+1}) \right] \quad (3) \end{aligned}$$

for all $(x_j, y_k) \in \Omega_h$.

Poisson eq.		Discrete Poisson eq.
$-\Delta u = f$ in Ω		$(L_h V)(x, y) = f(x, y) \quad (x, y) \in \Omega_h$
$u = g$ on $\partial\Omega$		$V(x, y) = g(x, y) \quad (x, y) \in \partial\Omega_h$

Stencil for scheme:



A grid function V is called discrete harmonic in Ω_n if

$$(L_h V)(x, y) = 0 \quad \forall (x, y) \in \Omega_n.$$

Theorem 2 (discrete max. principle)

If V is discrete harmonic, then

$$\min_{(x, y) \in \partial \Omega} V(x, y) \leq V_{j,k} \leq \max_{(x, y) \in \partial \Omega_n} V(x, y) \quad (4)$$

for all $(x_j, y_k) \in \Omega_n$.

Proof For any $(x_j, y_k) \in \Omega_n$,
 $(L_h V)(x_j, y_k) = 0 \xrightarrow{(3)} \Rightarrow$

$$V_{j,k} = \frac{V_{j-1,k} + V_{j+1,k} + V_{j,k-1} + V_{j,k+1}}{4}$$

$\Rightarrow$

$$\min(V_{j-1,k}, V_{j+1,k}, V_{j,k-1}, V_{j,k+1}) \leq V_{j,k} \leq \max(V_{j-1,k}, V_{j+1,k}, V_{j,k-1}, V_{j,k+1})$$

$\Rightarrow V$ has no strict maxima/minima in Ω_n

$\Rightarrow$ Max and min of V is on $\partial \Omega_n$. $\square$

Note: (4) implies that

$$\max_{(x,y) \in \Omega_n} |V(x,y)| \leq \max_{(x,y) \in \partial\Omega_n} |V(x,y)|.$$

Corollary 2 (discrete subharmonic)

If $L_n V \leq 0$ in Ω_n , then

$$V_{j,k} \leq \max_{(x,y) \in \partial\Omega_n} V(x,y)$$

Proof $(L_n V)(x_j, y_k) \leq 0$

$$\Rightarrow V_{j,k} \leq \frac{V_{j-1,k} + V_{j+1,k} + V_{j,k-1} + V_{j,k+1}}{4}.$$

and same argument as above. $\square$

Corollary 3 The discrete Poisson eq. has a unique solution.

Proof. Decompose grid functions $V = V^I + V^B$

where

$$V^I(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in \Omega_n \\ 0 & \text{if } (x,y) \in \partial\Omega_n \end{cases}$$

$$\& V^B(x,y) = \begin{cases} 0 & \text{if } (x,y) \in \Omega_n \\ V(x,y) & \text{if } (x,y) \in \partial\Omega_n \end{cases}$$

the discrete Poisson eq. becomes

$$V^B = g \quad \text{on } \partial\Omega_n \quad (5)$$

$$L_n(V^I) = \underbrace{f - L_n(V^B)}_{=:\hat{f} \text{ - given}} \quad \text{in } \Omega_n \quad (6)$$

(6) is a linear system of n^2 eq. and n^2 unknowns. The linear mapping $L_n: \mathbb{R}^{n^2} \rightarrow \mathbb{R}^{n^2}$ is invertible iff $\ker(L_n) = \{V^I \mid L_n(V^I) = 0 \text{ in } \Omega_n\} = \{0\}$.

$$V^I \in \ker(L_n) \Leftrightarrow L_n(V^I) = 0$$

$$\xRightarrow{\text{Thm 2}} \max_{\bar{\Omega}_n} |V^I| = \max_{\partial\Omega_n} |V^I| = 0.$$

$$\Rightarrow \ker(L_n) = \{0\}.$$



Lemma 1

Let v solve
$$\begin{cases} L_h v = q & \Omega_h \\ v = 0 & \partial\Omega_h \end{cases}$$

for some constant $q \in \mathbb{R}$, then

$$|V_{jk}| \leq \frac{|q|}{8}$$

Proof

$w(x,y) = \frac{q}{2} x(1-x)$ satisfies $L_h w = q$.

Note

$$L_h(w-v) = 0 \quad \Omega_h$$

$$w-v = \frac{q}{2} x(1-x) \quad \partial\Omega_h$$

Thm 2 $\Rightarrow$

$$\min\left(\frac{q}{8}, 0\right) \leq w-v \leq \max_{\partial\Omega_h} \frac{q}{2} x(1-x) = \max\left(\frac{q}{8}, 0\right)$$

If $q \geq 0$, then $w \geq 0$ and

$0 \leq w-v \leq \frac{q}{8}$ imply that

$$-\frac{q}{8} = \min_{\partial\Omega_h} w - \frac{q}{8} \leq v \leq \max_{\partial\Omega_h} w \leq \frac{q}{8}$$

If $q < 0$, then similarly, $\frac{q}{8} \leq v \leq -\frac{q}{8}$

~~QED~~

Theorem Let v solve $\begin{cases} L_h v = f & \Omega_h \\ v = 0 & \partial\Omega_h \end{cases}$

$$\text{Then } \|v\|_{h,\infty} \leq \frac{\|f\|_{h,\infty}}{8},$$

where $\|v\|_{h,\infty} := \max_{(x,y) \in \Omega_h} |v(x,y)|$.

Proof Let w solve $\begin{cases} L_h w = \|f\|_{h,\infty} & \Omega_h \\ w = 0 & \partial\Omega_h \end{cases}$

Then

$$\begin{aligned} L_h(v-w) &= f - \|f\|_{h,\infty} \leq 0 & \Omega_h \\ (v-w) &= 0 & \partial\Omega_h \end{aligned}$$

Corollary 2

$$\Rightarrow v-w \leq \max_{\partial\Omega_h} (v-w) = 0 \quad \text{in } \Omega_h$$

$$\Rightarrow v \leq w \leq \frac{\|f\|_{h,\infty}}{8} \quad \text{in } \Omega_h$$

↙ Lemma 1 ($q = \|f\|_{h,\infty}$)

Lower bound: Consider $-(v+w)$:

$$L_h(-(v+w)) = -(f + \|f\|_{h,\infty}) \leq 0 \quad \Omega_h$$

$$-(v+w) = 0 \quad \partial\Omega_h$$

$-(v+w)$ Subharmonic

$$\Rightarrow -(v+w) \leq \max_{\partial \Omega_n} (-(v+w)) = 0 \quad \text{in } \Omega_n$$

$$\Rightarrow v \geq -w \geq -\frac{\|f\|_{n,\infty}}{8} \quad \text{in } \Omega_n$$

□