

Lecture 15

Plan: Formal solutions of

$$\Delta u = 0 \quad \text{in } (x, y) \in \Omega$$

for 2D domains

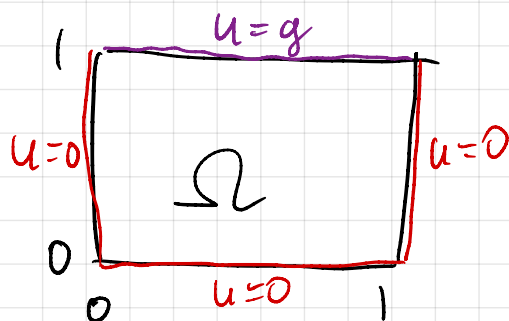
1. Box $\Omega =$ 

2. Disc $\Omega =$ 

3. Wedge $\Omega =$ 

Box problem

$$\left. \begin{aligned} \Delta u &= 0 \quad \text{in } \Omega = (0, 1)^2 \\ u(0, y) &= u(1, y) = 0 \quad 0 \leq y \leq 1 \\ u(x, 0) &= 0 \quad 0 \leq x \leq 1 \\ u(x, 1) &= g(x) \quad 0 < x < 1 \end{aligned} \right\} \begin{array}{l} (1) \\ (2) \end{array}$$



Ansatz: $u(x, y) = X(x) Y(y)$

Then $\Delta u = 0$

$$\Rightarrow X''(x)Y(y) + X(x)Y''(y) = 0 \quad | \frac{1}{XY}$$

$$\Rightarrow \frac{-X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} = \lambda \quad (3)$$

for some $\lambda \in \mathbb{R}$.

Eigenval problem

$$-X''(x) = \lambda X(x) \quad x \in (0, 1)$$

BC $X(0) = X(1) = 0$

(where $u(0, y) = X(0)Y(y) = 0 \quad \forall y \in [0, 1]$
 $\Rightarrow X(0) = 0$. $X(1) = 0$ obtained similarly.)

Solutions:

$$\lambda_k = (k\pi)^2 \quad \& \quad X_k(x) = \sin(k\pi x) \quad k \in \mathbb{N}$$

Second problem from (3):

$$Y_k''(y) = (k\pi)^2 Y_k(y) \quad 0 < y < 1$$

BC $Y_k(0) = 0$ (from $u(x, 0) = 0$)

General solution:

$$\Sigma_k(y) = a e^{k\pi y} + b e^{-k\pi y} \quad a, b \in \mathbb{R},$$

$$\& \Sigma(0) = 0 \Rightarrow a = -b$$

Setting $a = \frac{1}{2}$ yields

$$\Sigma_k(y) = \frac{e^{k\pi y} - e^{-k\pi y}}{2} = \sinh(k\pi y)$$

Particular solutions to (1):

$$u_k(x, y) = \underbrace{\sin(k\pi x)}_{\Sigma_k(x)} \underbrace{\sinh(k\pi y)}_{\Sigma_k(y)}, \quad k \in \mathbb{N}$$

How to adapt BC $u(x, 1) = g(x)$?

① Assume $g(x) = \sum_{k=1}^{\infty} g_k \sin(k\pi x)$.

② Formal solutions

$$u(x, y) = \sum_{k=1}^{\infty} c_k u_k(x, y) = \sum_{k=1}^{\infty} c_k \sinh(k\pi y) \sin(k\pi x)$$

$$\text{Then } u(x, 1) = g(x) \Rightarrow c_k = \frac{g_k}{\sinh(k\pi)}.$$

Disc domain

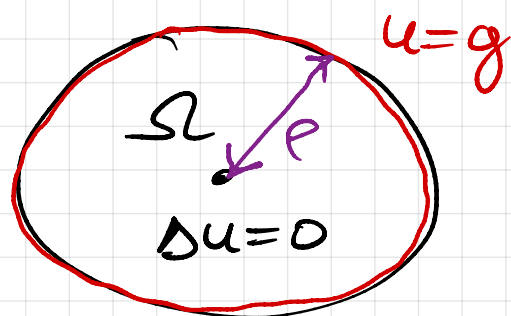
Consider $\Delta u(x,y) = 0$ in Ω (4)

$u(x,y) = g(x,y)$ on $\partial\Omega$ (5)

where $\Omega = \{(x,y) \mid \sqrt{x^2+y^2} < \rho\}$

$\partial\Omega = \{(x,y) \mid \sqrt{x^2+y^2} = \rho\}$

for some $\rho > 0$.

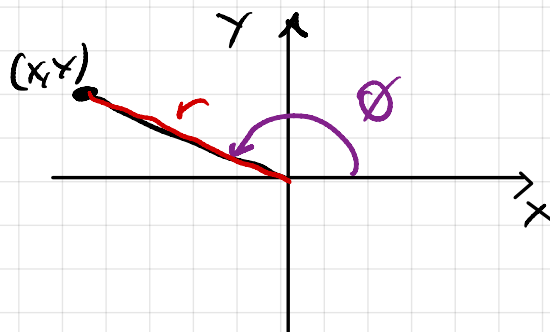


Easier to study (4) in polar coordinates:

$x = r \cos \theta$, $y = r \sin \theta$, $r \geq 0$, $\theta \in (-\pi, \pi]$

where $r = \sqrt{x^2 + y^2}$

$\theta = \arg(x + iy)$



Define $v(r,\theta) = u(x,y)$ ($= u(r \cos \theta, r \sin \theta)$)

PDE for v : know that

$$\Delta v = v_{xx} + v_{yy} = u_{xx} + u_{yy} = 0,$$

but how to express in $\frac{\partial}{\partial r}$ & $\frac{\partial}{\partial \phi}$?

$$V_x = V_r r_x + V_\phi \phi_x = V_r \cos \phi - V_\phi \frac{\sin \phi}{r}$$

since

$$r_x = \frac{\partial (x^2 + y^2)^{\frac{1}{2}}}{\partial x} = \frac{1}{2} \frac{2x}{\sqrt{x^2 + y^2}} = \frac{x}{r} = \cos \phi$$

and

$$y = r \sin \phi \quad \left| \frac{\partial}{\partial x} \right.$$

$$\Rightarrow 0 = r_x \sin \phi + r \cos \phi \phi_x$$
$$= \cos \phi (\sin \phi + r \phi_x)$$

$$\Rightarrow \phi_x = -\frac{\sin \phi}{r}$$

Using above and $r_y = \sin \phi$, $\phi_y = \frac{\cos \phi}{r}$

leads to $V_y = V_r \sin \phi + V_\phi \frac{\cos \phi}{r}$

and ...

$$V_{xx} + V_{yy} = V_{rr} + \frac{1}{r^2} V_{\phi\phi} + \frac{1}{r} V_r.$$

Eq (4) becomes

$$V_{rr} + \frac{1}{r^2} V_{\phi\phi} + \frac{1}{r} V_r = 0 \quad (r, \phi) \in [0, \rho) \times (-\pi, \pi) \quad (6)$$

$$BC: \quad \left. \begin{aligned} V(-\pi, r) &= V(\pi, r) \\ V_\phi(-\pi, r) &= V_\phi(\pi, r) \end{aligned} \right\} \quad 0 < r < \rho \quad (7)$$

Ex If $V = V(r)$ (rotation invariant),

then (6) becomes $V_{rr} + \frac{1}{r} V_r = 0$
or $V'' + \frac{1}{r} V' = 0$

Multiply by r :

$$(V' r)' = 0 \Rightarrow V' r = \text{constant}$$

$$\Rightarrow V(r) = c_1 \ln(r) + c_2$$

Separation of variables (6)-(7)

Guess $V(r, \phi) = R(r) \Phi(\phi)$

$$(6) \Rightarrow R'' \Phi + \frac{1}{r^2} R \Phi'' + \frac{1}{r} R' \Phi = 0 \quad | \quad \frac{r^2}{R \Phi}$$

Yields

$$\frac{r^2 R''(r) + r R'(r)}{R(r)} = \frac{-\Phi''(\phi)}{\Phi(\phi)} = \lambda \quad (8)$$

for some $\lambda \in \mathbb{R}$

Eigenval problem:

$$-\Phi''(\phi) = \lambda \Phi(\phi) \quad \phi \in (-\pi, \pi)$$

$$\text{BC} \quad \Phi(-\pi) = \Phi(\pi) \quad \& \quad \Phi'(-\pi) = \Phi'(\pi).$$

$$\text{Note:} \quad \int_{-\pi}^{\pi} -\Phi'' \Phi \, d\phi = \lambda \int_{-\pi}^{\pi} (\Phi)^2 \, d\phi$$
$$\int_{-\pi}^{\pi} (\Phi')^2 \, d\phi$$

$$\Rightarrow \lambda = \frac{\int_{-\pi}^{\pi} (\Phi')^2 d\phi}{\int_{-\pi}^{\pi} (\Phi)^2 d\phi} \geq 0$$

$\Rightarrow$ General solution

$$\Phi(\phi) = a \cos(\sqrt{\lambda} \phi) + b \sin(\sqrt{\lambda} \phi)$$

$$BC \Rightarrow \sqrt{\lambda} \in \{0, 1, 2, \dots\}$$

Eigenpairs:

$$\lambda_k = k^2, \quad \Phi_k(\phi) = \bar{a}_k \cos(k\phi) + \bar{b}_k \sin(k\phi)$$

Second problem in (8)

$$r^2 R_k'' + r R_k' - k^2 R_k = 0 \quad r \geq 0$$

with BC $R_k(0) = \begin{cases} \text{constant} & \text{if } k=0 \\ 0 & \text{if } k>0 \end{cases}$

(If $R_k(0) \neq 0$, then for some $\phi_1 \neq \phi_2$,

$$V_k(0, \phi_1) = R_k(0) \Phi_k(\phi_1) \neq R_k(0) \Phi_k(\phi_2) = V_k(0, \phi_2)$$

even if $(0, \phi_1)$ is same point as $(0, \phi_2)$.)

Insert guess solution $R_k(r) = r^\beta$ $\beta \geq 0$,
yields

$$\beta(\beta-1) \cancel{r^\beta} + \beta \cancel{r^\beta} - k^2 \cancel{r^\beta} = 0$$

$$\Rightarrow \beta_k = k \quad k = 0, 1, \dots$$

$$\text{Solution } R_k(r) = r^k \quad k = 0, 1, \dots$$

Particular solution

$$V_k(r, \vartheta) = r^k (\bar{a}_k \cos k\vartheta + \bar{b}_k \sin k\vartheta)$$

Formal solution

$$V(r, \vartheta) = \frac{\bar{a}_0}{2} + \sum_{k=1}^{\infty} r^k (\bar{a}_k \cos k\vartheta + \bar{b}_k \sin k\vartheta)$$

Adaptation for BC:

$$u = g \text{ on } \partial\Omega \Rightarrow V(\rho, \vartheta) = g(\vartheta) \quad \vartheta \in (-\pi, \pi]$$

$$\text{If } g(\vartheta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\vartheta) + b_k \sin(k\vartheta)$$

$$\text{then } V(\rho, \vartheta) = g(\vartheta) \Rightarrow \bar{a}_0 = a_0$$

$$\left. \begin{aligned} \bar{a}_k &= \rho^{-k} a_k \\ \bar{b}_k &= \rho^{-k} b_k \end{aligned} \right\} \quad k \geq 1$$

Example

$$u_{xx}(r, \vartheta) + u_{yy}(r, \vartheta) = 0 \text{ in } \Omega = \{(x, y) \mid \sqrt{x^2 + y^2} < 2\}$$

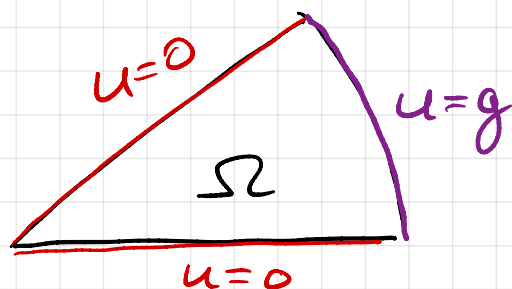
$$\text{BC: } u(2, \vartheta) = \underbrace{1 + 3\cos 2\vartheta}_{=g(\vartheta)}.$$

$a_0 = 2, a_2 = 3$ & $\rho = 2$ yields solution

$$u(r, \vartheta) = 1 + \frac{3}{4} r^2 \cos(2\vartheta)$$

Wedge domain

$\Delta u = 0$ in $\Omega = \{(r, \varphi) \mid 0 < r < \rho, 0 < \varphi < \frac{\pi}{b}\}$
for some $b > \frac{1}{2}$.



$$\begin{aligned} \text{BC } u(r, 0) &= u(r, \frac{\pi}{b}) = 0 & 0 \leq r \leq \rho \\ u(\rho, \varphi) &= g(\varphi) & 0 < \varphi < \frac{\pi}{b} \end{aligned}$$

Guess solution

$$u(r, \varphi) = R(r) \Phi(\varphi)$$

leads again to (8), Problem

$$-\Phi_k'' = \lambda_k \Phi_k$$

$$\Phi_k(0) = \Phi_k(\frac{\pi}{b}) = 0$$

with solution $\Phi_k(\varphi) = \sin(kb\varphi)$

$$\lambda_k = (kb)^2 \quad k = 0, 1, \dots$$

$$\text{and } r^2 R_k''(r) + r R_k'(r) - k^2 b^2 R_k(r) = 0 \quad r > 0$$

$$\text{with BC } R_k(0) = 0$$

has solution

$$R_k(r) = \begin{cases} 0 & k=0 \\ r^{bk} & k \geq 1 \end{cases}$$

Particular solutions:

$$u_k(r, \varphi) = r^{bk} \sin(bk\varphi) \quad k=1, 2, \dots$$

Formal solution:

$$u(r, \varphi) = \sum_{k=1}^{\infty} c_k r^{bk} \sin(bk\varphi)$$

$$\text{If } g(\varphi) = \sum_{k=1}^{\infty} a_k \sin(bk\varphi),$$

$$\text{then } u(\rho, \varphi) = g(\varphi) \Rightarrow c_k = \rho^{-bk} a_k.$$

NB! (Solution u_1 has low regularity:)

$$\begin{aligned} \frac{\partial}{\partial r} u_1(r, \varphi) &= \frac{\partial}{\partial r} r^b \sin(b\varphi) \\ &= b r^{b-1} \sin(b\varphi) \end{aligned}$$

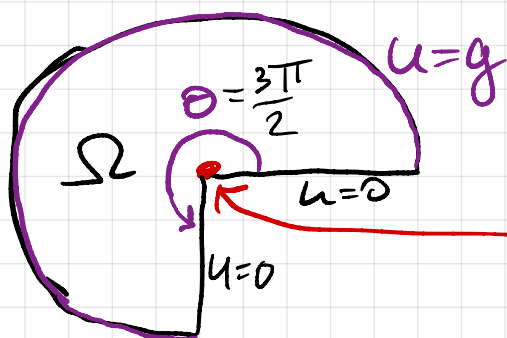
So if $b < 1$, then

$$\lim_{r \downarrow 0} \frac{\partial}{\partial r} u_1(r, \frac{\pi}{2}) = \infty$$

Ex $p=1$, $b=\frac{2}{3}$ & $g(\varnothing) = \sin(\frac{2}{3}\varnothing)$

gives solution

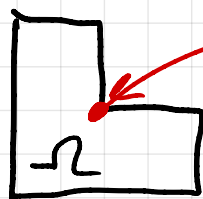
$$u(r, \varnothing) = r^{2/3} \sin(\frac{2}{3}\varnothing) \quad 0 \leq r \leq 1, 0 \leq \varnothing \leq \frac{3}{2}\pi$$



Corner singularity

For domains with $\theta = \frac{\pi}{b} > 1$ ($b < 1$),
PDE $\Delta u = 0$ admits corner
singularities.

Also problem in L-shaped domains:



Corner singularity
for some harmonic $u(x,y)$.