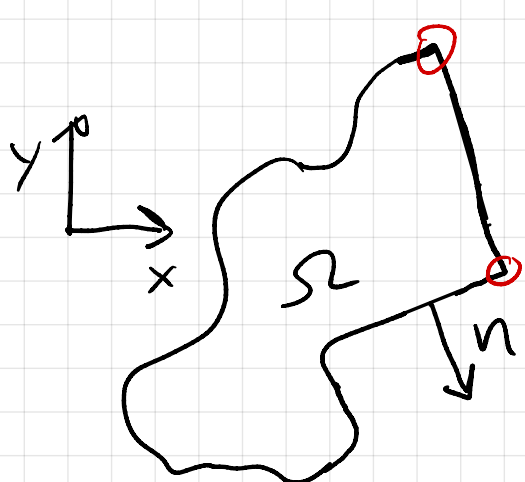


Lecture 16

Let $\Omega \subset \mathbb{R}^2$ be open, bounded and connected with piecewise C^1 boundary $\partial\Omega$.
Unit normal $n = (n_1, n_2)$
(defined everywhere except points where boundary is not C^1)



For any (smooth) $f: \Omega \rightarrow \mathbb{R}$,

$$\int_{\Omega} f_x \, dx dy = \int_{\partial\Omega} f n_1 \, ds$$

$$\int_{\Omega} f_y \, dx dy = \int_{\partial\Omega} f n_2 \, ds$$

$\Rightarrow$ Divergence theorem:

For any (smooth) vector field $F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}: \Omega \rightarrow \mathbb{R}^2$

$$\int_{\Omega} \underbrace{\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}}_{\operatorname{div} F} \, dx dy = \int_{\partial\Omega} \underbrace{f_1 n_1 + f_2 n_2}_{F \cdot n} \, ds$$

Consequences:-

• For $F = \nabla u = \begin{pmatrix} u_x \\ u_y \end{pmatrix}$ this gives

$$\operatorname{div} F = \nabla \cdot \nabla u = \Delta u, \quad F \cdot n = \nabla u \cdot n =: \frac{\partial u}{\partial n}$$

$$\int_{\Omega} \Delta u \, dx dy = \int_{\partial \Omega} \frac{\partial u}{\partial n} \, ds$$

• For $F = v \nabla u$, $\left(\begin{array}{l} v: \Omega \rightarrow \mathbb{R} \\ u: \Omega \rightarrow \mathbb{R} \end{array} \right)$

$$\operatorname{div} F = \nabla \cdot (v \nabla u) = \nabla v \cdot \nabla u + v \Delta u \quad \text{and}$$

$$F \cdot n = v(\nabla u \cdot n) = v \frac{\partial u}{\partial n}$$

$$\int_{\Omega} v \Delta u + \nabla v \cdot \nabla u \, dx dy = \int_{\partial \Omega} v \frac{\partial u}{\partial n} \, ds \quad (1)$$

For $v \equiv 1$, (1) yields

$$\int_{\Omega} \Delta u \, dx dy = \int_{\partial \Omega} \frac{\partial u}{\partial n} \, ds$$

Note: if $\Delta u = 0$ in Ω , then

$$\int_{\partial \Omega} \frac{\partial u}{\partial n} \, ds = 0 \quad (2)$$

For $v = u$, (1) yields

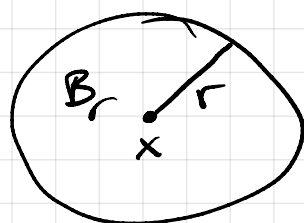
$$\int_{\Omega} u \Delta u + |\nabla u|^2 \, dx dy = \int_{\partial \Omega} u \frac{\partial u}{\partial n} \, ds$$

Mean value property

Notation: $x, y, z \in \mathbb{R}^2$, $x = (x_1, x_2)$ etc.

$$B_r(x) = \{y \in \mathbb{R}^2 \mid |x - y| < r\} \quad \text{open disc}$$

$\uparrow$ center $\uparrow$ radius



$$\partial B_r = \{y \in \mathbb{R}^2 \mid |x - y| = r\}$$

Thm (mean value prop)

If u is harmonic in Ω ($\Delta u = 0$ in Ω),
then for any $B_r(x) \subset \Omega$,

$$\underbrace{u(x)}_{\substack{\text{val of} \\ u \text{ at point} \\ x}} = \underbrace{\frac{1}{2\pi r} \int_{\partial B_r(x)} u(y) dS(y)}_{\text{avg value of } u \text{ on } \partial B_r(x)} \quad (*)$$

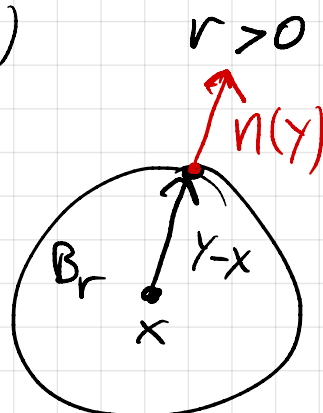
(Comment, opposite also true: for $u \in C^2(\Omega)$, $(*) \Rightarrow \Delta u = 0$.)

Proof Define

$$U(r) := \frac{1}{2\pi r} \int_{\partial B_r(x)} u(y) dS(y)$$

Note that for

$$y \in \partial B_r(x), \quad n(y) = \frac{y - x}{r}$$



Change of var :

$$Y = x + rz, \quad z \in \partial B_1(0) :$$

$$\Rightarrow dS(Y) = r dS(z)$$

$$\underline{U}(r) = \frac{1}{2\pi} \int_{\partial B_1(0)} u(x + rz) dS(z)$$

So

$$\underline{U}'(r) = \frac{1}{2\pi} \int_{\partial B_1(0)} \frac{d}{dr} u(x + rz) dS(z)$$

$$= \frac{1}{2\pi} \int_{\partial B_1(0)} \nabla u(x + rz) \cdot z dS(z)$$

$$= \frac{1}{2\pi r} \int_{\partial B_r(x)} \nabla u(y) \cdot \underbrace{\frac{y-x}{r}}_{n(y)} dS(y)$$

$$= \frac{1}{2\pi r} \int_{B_r(x)} \Delta u(y) dy = 0.$$

$$\Rightarrow \underline{U}(r) = \lim_{\bar{r} \rightarrow 0} \underline{U}(\bar{r})$$

$$= \lim_{\bar{r} \rightarrow 0} \frac{1}{2\pi} \int_{\partial B_1(0)} u(x + \bar{r}z) dS(z)$$

$$= u(x) + \frac{1}{2\pi} \int_{\partial B_1(0)} \lim_{\bar{r} \rightarrow 0} (u(x + \bar{r}z) - u(x)) dS(z) = u(x)$$

Corollary

If u is harmonic in Ω , then

$$u(x) = \frac{1}{\pi r^2} \int_{B_r(x)} u(y) dy$$

for any $B_r(x) \subset \Omega$

Proof: Change of variables

$$B_r(x) \ni y = x + (p \cos \theta, p \sin \theta), \quad 0 \leq p \leq r, \quad \theta \in [0, 2\pi)$$

$$\Rightarrow \int_{B_r(x)} u(y) dy = \int_0^r \int_0^{2\pi} \underbrace{u(x + (p \cos \theta, p \sin \theta)) d\theta p}_{= 2\pi U(p) = 2\pi u(x)} dp$$

$$= 2\pi \int_0^r u(x) p dp = \pi r^2 u(x) \quad \square$$

Properties of operator $-\Delta$

Let $C_0^2(\Omega) := \{u \in C^2(\Omega) \cap C(\bar{\Omega}) \mid u|_{\partial\Omega} = 0\}$,

We define on $C_0^2(\Omega)$:

- inner product $\langle u, v \rangle = \int_{\Omega} u(x) v(x) dx$

- and operator L by $Lu = -\Delta u$

Properties

- Symmetric

$$\langle Lu, v \rangle = \langle u, Lv \rangle \quad \forall u, v \in C_0^2(\Omega)$$

$$\begin{aligned} \text{As } \langle Lu, v \rangle &= \int_{\Omega} (-\Delta u) v \, dx \\ &= \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \underbrace{v}_{=0} \frac{\partial u}{\partial n} \, ds \\ &= \int_{\Omega} u (-\Delta v) \, dx \end{aligned}$$

- Positive definite:

$$\langle Lu, u \rangle \geq 0 \quad \forall u \in C_0^2(\Omega)$$

$$\& \langle Lu, u \rangle = 0 \quad \Leftrightarrow \quad u \equiv 0$$

$$\begin{aligned} \text{As } \langle Lu, u \rangle &= \int_{\Omega} (-\Delta u) u \, dx = \int |\nabla u|^2 \, dx \\ &\geq 0 \end{aligned}$$

and if $\nabla u \equiv 0$, then $u = \text{constant} = 0$ as $u|_{\partial\Omega} = 0$.

Theorem

The Poisson eq. $Lu = f$ in Ω with Dirichlet bd $u = 0$ on $\partial\Omega$ has at most one sol in $C_0^2(\Omega)$

Proof

Assume $u, v \in C_0^2(\Omega)$ both are solutions,
meaning $Lu = f$ & $Lv = f$.

Then $L(u-v) = 0$ and $(u-v) \in C_0^2(\Omega)$
implies that $\underbrace{\langle L(u-v), (u-v) \rangle}_{=0} = 0,$

hence $u \equiv v$.

