

Lecture 17

Properties of L_h in 2D

Continuous domain	Discrete domain
	$h = \frac{1}{n+1}$ & points
	$(x_j, y_k) = (jh, kh)$
$\Omega = \{(x, y) \mid 0 < x, y < 1\}$	$\Omega_h = \{(x_j, y_k) \mid 1 \leq j, k \leq n\}$
$\bar{\Omega} = \{(x, y) \mid 0 \leq x, y \leq 1\}$	$\bar{\Omega}_h = \{(x_j, y_k) \mid 0 \leq j, k \leq n+1\}$
$\partial\bar{\Omega} = \bar{\Omega} \setminus \Omega$	$\partial\bar{\Omega}_h = \bar{\Omega}_h \setminus \Omega_h$

Poisson's problem:	Discrete problem
Find $u \in C_0^2(\Omega)$ s.t.	Find $v \in D_{h,0}$ s.t.
$Lu = f$ in Ω (1)	$(L_h v)(x_j, y_k) = f(x_j, y_k)$ in Ω_h (2)

where $D_{h,0} = \{v: \bar{\Omega}_h \rightarrow \mathbb{R} \mid v|_{\partial\bar{\Omega}_h} = 0\}$
and

$$\begin{aligned}(L_h v)(x_j, y_k) &= h^{-2} \left[4v_{j,k} - (v_{j+1,k} + v_{j-1,k} + v_{j,k+1} + v_{j,k-1}) \right] \\ &= h^{-2} \left[- (v_{j+1,k} - v_{j,k}) + (v_{j,k} - v_{j-1,k}) - (v_{j,k+1} - v_{j,k}) + (v_{j,k} - v_{j,k-1}) \right]\end{aligned}$$

(Goals: ① L_h symmetric and pos def on $D_{h,0}$,
② conv. rate of $v_{j,k} \rightarrow u(x_j, y_k)$ as $h \rightarrow 0$.)

Inner product on $D_{h,0}$:

$$\langle u, v \rangle_n := h^2 \sum_{j,k=1}^n u_{j,k} v_{j,k} \quad u, v \in D_{h,0}.$$

Properties: For any $u, v \in D_{h,0}$

$$\langle u, v \rangle_n = \langle v, u \rangle_n \quad (3)$$

$$\begin{aligned} \sum_{j=1}^n u_{j-1,k} v_{j,k} &= u_{0,k} v_{1,k} + u_{1,k} v_{2,k} + \dots + u_{n-1,k} v_{n,k} + \underbrace{u_{n,k} v_{n+1,k}}_{=0} \\ &= \sum_{j=0}^n v_{j+1,k} u_{j,k} \end{aligned}$$

$$\text{and } \sum_{j=1}^n v_{j,k} u_{j,k} = \sum_{j=0}^n v_{j+1,k} u_{j+1,k}$$

$$\Rightarrow \sum_{j=1}^n (u_{j,k} - u_{j-1,k}) v_{j,k} = \sum_{j=0}^n (u_{j+1,k} - u_{j,k}) v_{j+1,k} \quad (4)$$

and, similarly,

$$\sum_{k=1}^n (u_{j,k} - u_{j,k-1}) v_{j,k} = \sum_{k=0}^n (u_{j,k+1} - u_{j,k}) v_{j,k+1} \quad (5)$$

Lemma 1

a) The operator L_h is symmetric on $D_{h,0}$:

$$\langle L_h u, v \rangle_n = \langle u, L_h v \rangle_n \quad \forall u, v \in D_{h,0}$$

b) and L_h is positive definite on $D_{h,0}$:

$$\langle L_h u, u \rangle_n \geq 0 \quad \forall u \in D_{h,0} \text{ with}$$

$$\langle L_h u, u \rangle_n = 0 \Leftrightarrow u \equiv 0.$$

Proof

$$\begin{aligned} a) \langle L_n u, v \rangle_n &= \sum_{k=1}^n \sum_{j=1}^n (u_{j+1,k} - u_{j,k}) (-v_{j,k}) + (u_{j,k} - u_{j-1,k}) v_{j,k} \\ &\quad + \sum_{j=1}^n \sum_{k=1}^n (u_{j,k+1} - u_{j,k}) (-v_{j,k}) + (u_{j,k} - u_{j,k-1}) v_{j,k} \\ &\stackrel{(4)(5)}{=} \sum_{j,k=0}^n (u_{j+1,k} - u_{j,k}) (v_{j+1,k} - v_{j,k}) + (u_{j,k+1} - u_{j,k}) (v_{j,k+1} - v_{j,k}) \end{aligned}$$

By direct calculation, also

$$\langle L_n v, u \rangle_n = \sum_{j,k=0}^n (u_{j+1,k} - u_{j,k}) (v_{j+1,k} - v_{j,k}) + (u_{j,k+1} - u_{j,k}) (v_{j,k+1} - v_{j,k})$$

$$\text{Hence } \langle L_n u, v \rangle_n = \langle L_n v, u \rangle_n \stackrel{(3)}{=} \langle u, L_n v \rangle$$

b)

$$\langle L_n u, u \rangle_n = \sum_{j,k=0}^n (u_{j+1,k} - u_{j,k})^2 + (u_{j,k+1} - u_{j,k})^2 \geq 0,$$

and

$$\langle L_n u, u \rangle = 0 \Rightarrow u_{j,k} = u_{j+1,k} = u_{j,k+1} \quad (1 \leq j, k \leq n)$$

$$\Rightarrow u \equiv \text{constant and} \\ \text{constant} = 0 \text{ as } u_{j,0} = 0$$

□

Error estimate

Let $u \in C^2_0(\Omega) \cap C^4(\bar{\Omega})$ solve

$$Lu = f \quad \text{in } \Omega$$

and let $v \in D_{h,0}$ solve

$$L_h v = f \quad \text{in } \Omega_h \quad \text{for given } h > 0.$$

Consider truncation error

$$T_h(x_j, y_k) = (L_h u - f)(x_j, y_k)$$

$$= \frac{u_{xx}(x_j, y_k) h^2 + u_{yy}(x_j, y_k) h^2}{h^2} - f(x_j, y_k) + \mathcal{O}(h^2)$$

It holds that

$$\max_{(x,y) \in \Omega_h} |T_h(x,y)| \leq \frac{\alpha h^2}{6},$$

where

$$\alpha = \max \left(\max_{(x,y) \in \bar{\Omega}} \left| \frac{\partial^4}{\partial x^4} u(x,y) \right|, \max_{(x,y) \in \bar{\Omega}} \left| \frac{\partial^4}{\partial y^4} u(x,y) \right| \right)$$

and

$$\max_{(x,y) \in \bar{\Omega}_h} |u(x,y) - v(x,y)| \leq \frac{\alpha h^2}{48} \quad (6)$$

Proof of (6):

$$\text{Let } e(x_j, y_k) := u(x_j, y_k) - v(x_j, y_k) \quad (x_j, y_k) \in \overline{\Omega}_h$$

Then $e \in D_{h,0}$, and e solves

$$L_h e = L_h u - L_h v = L_h u - f = T_h \quad \text{in } \Omega_h.$$

By discrete max-principle (Lecture 14),

$$\|e\|_{h,\infty} \leq \frac{\|T_h\|_{h,\infty}}{8} \leq \frac{\alpha h^2}{48}.$$

□

Fourier series on $[-1, 1]$:

Recall $\{\cos(k\pi x)\}_{k=0}^{\infty} \cup \{\sin(k\pi x)\}_{k=1}^{\infty}$ is orthogonal with respect to inner product

$$\langle u, v \rangle = \int_{-1}^1 u(x)v(x) dx \quad (\text{Exercise 3.15})$$

and Fourier series of $f: [-1, 1] \rightarrow \mathbb{R}$ is

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(k\pi x) + b_k \sin(k\pi x))$$

$$\text{where } a_k = \langle f, \cos(k\pi x) \rangle$$

$$b_k = \langle f, \sin(k\pi x) \rangle$$

" $\sim$ " will denote "Fourier series of f equals".

Truncated Fourier series

$$S_N(f)(x) := \frac{a_0}{2} + \sum_{k=1}^N (a_k \cos(k\pi x) + b_k \sin(k\pi x))$$

Note $f \sim S_{\infty}(f)$ and it does not always hold that $f(x) = S_{\infty}(f)(x)$:

Ex For $f(x) = x$, one can show that

$$a_k = \int_{-1}^1 \cos(k\pi x) x dx = 0 \quad \&$$

$$b_k = \int_{-1}^1 \sin(k\pi x) x dx = \frac{2}{k\pi} (-1)^{k+1} \quad \text{yielding}$$

$$f(x) \sim \sum_{k=1}^{\infty} \frac{2}{k\pi} (-1)^{k+1} \sin(k\pi x) = S_{\infty}(f)(x)$$

Note that $f(-1) = -1$, while

$$S_{\infty}(f)(1) = \sum_{k=1}^{\infty} \frac{2}{k\pi} (-1)^{k+1} \cdot \underbrace{\sin(0)}_{=0} = 0$$

Will nevertheless later show that

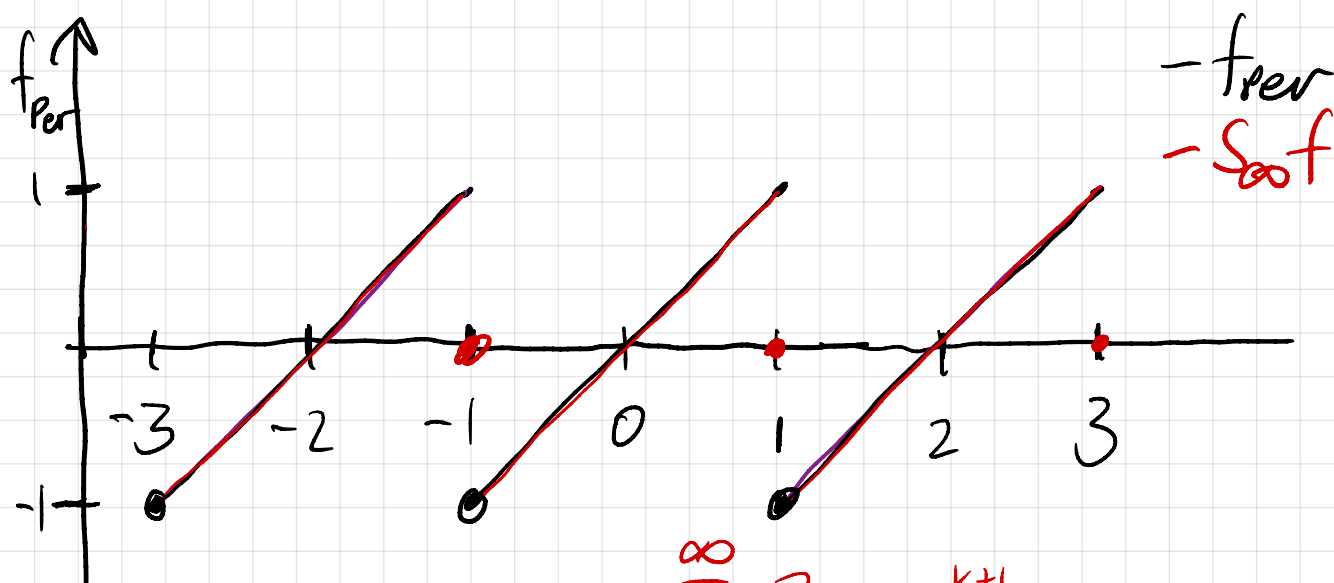
$$\int_{-1}^1 (f(x) - S_N(f)(x))^2 dx \rightarrow 0 \text{ as } N \rightarrow \infty.$$

A clue to why $f(1) \neq S_{\infty}(1)$ follows from periodic extensions:

Any function $f: [-1, 1)$ has a unique 2-periodic extension f_{per} on $\mathbb{R}$ s. t.

$$f_{\text{per}}(x+2) = f_{\text{per}}(x) \quad \forall x \in \mathbb{R}$$

For $f(x) = x$, we get



while $S_{\infty}(f)(x) = \sum_{k=1}^{\infty} \frac{2}{k\pi} (-1)^{k+1} \sin(k\pi x)$ is 2-periodic by construction.

And $S_{\infty}(f)(-1) = \frac{f(-1) + f(1)}{2} \neq f(-1)$ when $f(-1) \neq f(1)$.

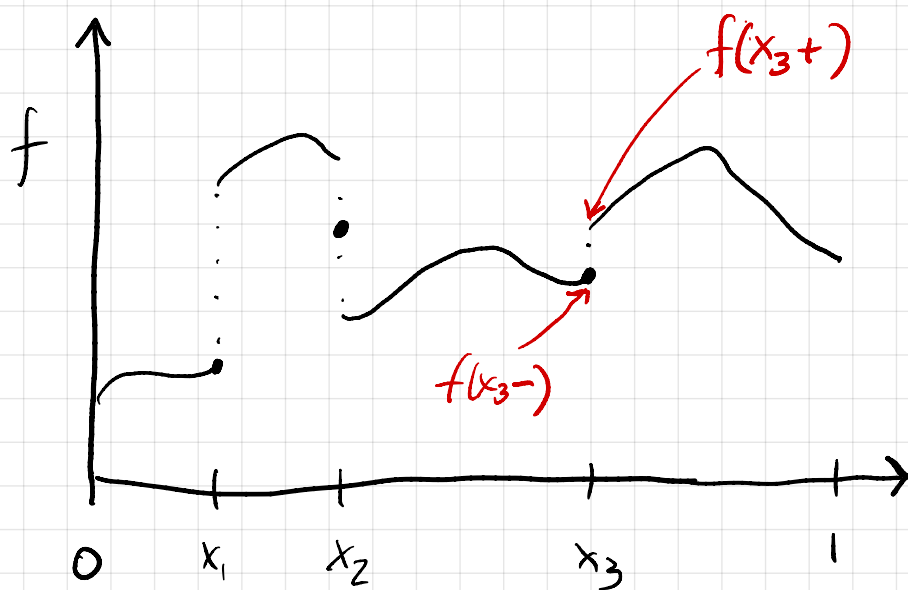
Goals today:

- Properties of $S_\infty(t)$: even/odd parts, differentiation
- Next lectures: mixed boundary val problems,
convergence $S_N(t) \rightarrow f$ (when and how does it hold?)

We assume the following regularity for f :

Definition f on $[-1, 1]$ is piecewise continuous if it is continuous at all but a finite number of points $\{x_i\}$, and all limits

$$\lim_{x \uparrow x_i} f(x) =: f(x_i-) \quad \& \quad \lim_{x \downarrow x_i} f(x) =: f(x_i+) \text{ exist.}$$



f is called even if $f(x) = f(-x)$
and odd if $f(x) = -f(-x) \quad \forall x \in [-1, 1]$

Ex even: $1, x^2, x^4, \dots, \cos(k\pi x)$
odd: $x, x^3, \sin(k\pi x)$

Property If f is odd, then

$$f(x) \sim \sum_{k=1}^{\infty} b_k \sin(k\pi x)$$

and if f is even, then

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\pi x)$$

Proof odd f :

$$\begin{aligned} a_k &= \int_{-1}^1 f(x) \cos(k\pi x) dx \\ &= \int_{-1}^0 f(x) \cos(k\pi x) dx + \int_0^1 f(x) \cos(k\pi x) dx \quad (y = -x) \\ &= \int_0^1 \underbrace{f(-y) \cos(-k\pi y)}_{= -f(y) \cos(k\pi y)} dy + \int_0^1 f(x) \cos(k\pi x) dx \\ &= 0 \end{aligned}$$

For a function $\tilde{f}$ on $[0,1]$ with sine series

$$\sum_{k=1}^{\infty} \tilde{b}_k \sin(k\pi x), \quad \tilde{b}_k = 2 \int_0^1 \tilde{f}(x) \sin(k\pi x) dx,$$

with odd extension on $[-1,1]$:

$$f(x) := \begin{cases} \tilde{f}(x) & x \in [0,1] \\ -\tilde{f}(-x) & x \in [-1,0) \end{cases}$$

Then f is odd and

$$S_{\infty}(f) = \sum_{k=1}^{\infty} b_k \sin(k\pi x) \quad \text{with } b_k = \tilde{b}_k \text{ as}$$

$$b_k = \int_{-1}^1 f(x) \sin(k\pi x) dx = 2 \int_0^1 \tilde{f}(x) \sin(k\pi x) dx$$

$$\Rightarrow S_{\infty}(f) = \text{sine series of } \tilde{f} = f|_{[0,1]} \quad (\text{when } f \text{ is odd})$$

$$\text{Application: Let } f = \text{sign}(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

Then f is odd extension of

$$f|_{[0,1]} \equiv 1 \quad x \in [0,1]$$

$$1 \text{ has sine series } \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)} \sin((2k-1)\pi x) \quad x \in [0,1]$$

$$\Rightarrow \text{sign}(x) \sim \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)} \sin((2k-1)\pi x) \quad x \in [-1,1]$$

Similarly, if f is even, then
 $S_\infty(f) = \text{cosine series of } f|_{[0,1]}$.

Theorem (differentiation of Fourier series)

Assume that $f \in C([-1, 1])$, f' is piecewise cont. on $[-1, 1]$, and $f(-1) = f(1)$. Let

$$S_\infty(f)(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(k\pi x) + b_k \sin(k\pi x))$$

and

$$S_\infty(f')(x) = \frac{\alpha_0}{2} + \sum_{k=1}^{\infty} (\alpha_k \cos(k\pi x) + \beta_k \sin(k\pi x))$$

Then $\alpha_0 = 0$ and $\alpha_k = k\pi b_k$
 $\beta_k = -k\pi a_k$ $\left\{ k \geq 1 \right\}$.

Means that $S_\infty(f)'(x) = S_\infty(f')(x)$

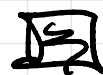
Proof $\alpha_0 = \int_{-1}^1 f'(x) dx = f(1) - f(-1) = 0,$

$$\alpha_k = \int_{-1}^1 f'(x) \cos(k\pi x) dx$$

$$= \left[f(x) \cos(k\pi x) \right]_{-1}^1 + k\pi \int_{-1}^1 f(x) \sin(k\pi x) dx$$

$$= \underbrace{f(1)^k (f(1) - f(-1))}_{=0} + k\pi b_k$$

β_k is computed similarly.



Application

For $f(x) = |x|$, let us find $S_\infty(f)$.

Since $f'(x) = \text{sign}(x)$, we know that

$$S_\infty(f')(x) = \sum_{k=1}^{\infty} \underbrace{\frac{4}{\pi(2k-1)}}_{\beta_{2k-1}} \sin((2k-1)\pi x)$$

$$\text{Then } a_{2k-1} = -\frac{\beta_{2k-1}}{\pi(2k-1)} = -\frac{4}{\pi^2(2k-1)^2}$$

$$\text{and } a_0 = \int_{-1}^1 |x| dx = 1$$

$$\Rightarrow S_\infty(f)(x) = \frac{1}{2} - \sum_{k=1}^{\infty} \frac{4}{\pi^2(2k-1)^2} \cos((2k-1)\pi x)$$