

Lecture 18

Plan: General Fourier series &
Sturm-Liouville problems

$$\text{For } \left. \begin{array}{l} u_t = u_{xx} \quad x \in (0,1), t > 0 \\ u(t,0) = u(t,1) = 0 \quad t \geq 0 \end{array} \right\}$$

Ansatz $u(x,t) = \underline{X}(x)T(t)$ leads to
eigenval problem (that we can solve)

$$\begin{aligned} -\underline{X}''(x) &= \lambda \underline{X}(x) \quad x \in (0,1) \\ \underline{X}(0) &= \underline{X}(1) = 0 \end{aligned}$$

Extension to other Dirichlet BC:

$$\begin{aligned} v_t &= v_{xx} \\ v(0,t) &= a, \quad v(1,t) = b \end{aligned}$$

$\Rightarrow v(t,x) = u(t,x) + a(1-x) + bx$ is solution
(See Exercise 3.11)

For mixed BC

$$\left. \begin{array}{l} u_t = u_{xx} \\ u(0,t) = 0, \quad u_x(1,t) = 0 \quad t > 0, \end{array} \right\} (2)$$

leads to eigenvalue problem

$$\left. \begin{array}{l} \underbrace{-\underline{X}''(x)}_{=:\underline{L}\underline{X}} = \lambda \underline{X}(x) \quad x \in (0,1) \\ \underline{X}(0) = 0, \quad \underline{X}'(1) = 0 \end{array} \right\} (3)$$

Generalization of Fourier series: we seek orthogonal nonzero sequence $\bar{X}_k(x)$ $k=0,1,\dots$ solving (3) (being eigen functions of L),

That is: $\langle \bar{X}_k, \bar{X}_m \rangle = \int_0^1 \bar{X}_k(x) \bar{X}_m(x) dx = 0$ if $k \neq m$

First observe $\int_0^1 -\bar{X}''(x) \bar{X}(x) dx = \lambda \int_0^1 (\bar{X}(x))^2 dx$
 $\int_0^1 (\bar{X}'(x))^2 dx$

$$\bar{X} \neq 0 \text{ \& } \bar{X}(0)=0 \Rightarrow \bar{X}' \neq 0 \Rightarrow \lambda = \frac{\int_0^1 (\bar{X}')^2 dx}{\int_0^1 (\bar{X})^2 dx} > 0.$$

General sol: $\bar{X}(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$

$$\bar{X}(0)=0 \Rightarrow c_1=0$$

$$0 = \bar{X}'(1) = c_2 \sqrt{\lambda} \cos(\sqrt{\lambda})$$

Set $c_2=1$ (any value $\neq 0$ is ok). Then

$$\cos(\sqrt{\lambda})=0 \Rightarrow \sqrt{\lambda} = (k + \frac{1}{2})\pi, k=0,1,\dots$$

And $\lambda_k = (k + \frac{1}{2})^2 \pi^2$, $\bar{X}_k(x) = \sin((k + \frac{1}{2})\pi x)$

are orthogonal eigenfunctions of L :

$$\begin{aligned} \lambda_k \langle \bar{X}_k, \bar{X}_m \rangle &= \langle L \bar{X}_k, \bar{X}_m \rangle \\ &= \langle \bar{X}_k, L \bar{X}_m \rangle \quad (\text{integration by parts}) \\ &= \lambda_m \langle \bar{X}_k, \bar{X}_m \rangle, \end{aligned}$$

$$\lambda_k \neq \lambda_m \Rightarrow \langle \bar{x}_k, \bar{x}_m \rangle = 0.$$

$$\begin{aligned} (\langle L\bar{x}_k, \bar{x}_m \rangle &= \int_0^1 -\bar{x}_k'' \bar{x}_m dx \\ &= [-\bar{x}_k' \bar{x}_m]_0^1 + \int_0^1 \bar{x}_k' \bar{x}_m' dx \\ &= \int_0^1 \bar{x}_k (-\bar{x}_m'') dx + [\bar{x}_k \bar{x}_m']_0^1 \\ &= \langle \bar{x}_k, L\bar{x}_m \rangle) \end{aligned}$$

Particular solutions of (2)

$$\begin{aligned} u_k(x, t) &= T_k(t) \bar{x}_k(x) \\ &= e^{-(k+\frac{1}{2})^2 \pi^2 t} \sin((k+\frac{1}{2})\pi x) \quad k=0, 1, \dots \end{aligned}$$

(since $T_k' = -\lambda_k T_k$ has sol $T_k(t) = e^{-\lambda_k t}$)

$$\text{Formal sol of (2): } u(x, t) = \sum_{k=0}^{\infty} c_k u_k(x, t)$$

where c_k is determined from boundary data. If $u(x, 0) = f(x)$, for example, then

$$c_k = \frac{\langle f, \bar{x}_k \rangle}{\langle \bar{x}_k, \bar{x}_k \rangle}.$$

See Ex 8.5 in TW for solving

$$-\mathcal{X}'' = \lambda \mathcal{X} \quad \text{with robin BC}$$

$$\mathcal{X}'(0) = \mathcal{X}(0), \quad \mathcal{X}'(1) = -\mathcal{X}(1).$$

Sturm-Liouville problems

Consider extension of heat eq:

$$\left. \begin{aligned} u_t &= (p u_x)_x - q u & x \in (0,1), t > 0 \\ u(0,t) &= u(1,t) = 0 & t \geq 0 \\ u(x,0) &= f(x) & x \in (0,1) \end{aligned} \right\} (4)$$

for known functions

$$\left. \begin{aligned} p &= p(x) \geq \alpha > 0 \\ q &= q(x) \geq 0 \end{aligned} \quad \forall x \in [0,1] \quad \text{with } p', q \in C([0,1]) \right\} (5)$$

Obtain particular solutions of (4) by ansatz $u(x,t) = T(t)\mathcal{X}(x)$

$$\Rightarrow \frac{-T'(t)}{T(t)} = \frac{-(p(x)\mathcal{X}')'}{\mathcal{X}(x)} + q(x) = \lambda$$

Eigenval problem: seek $\mathcal{X} \in C_0^2((0,1)) \setminus \{0\}$ s.t.

$$\underbrace{-(p\mathcal{X}')'}_{=: L\mathcal{X}} + q\mathcal{X} = \lambda \mathcal{X} \quad \forall x \in (0,1) \quad (6)$$

(6) is called a Sturm-Liouville problem (with Dirichlet 0-valued BC).

It holds $\forall x, y \in C_0^2(0,1)$ that

$$(a) \quad \langle Lx, y \rangle = \langle x, Ly \rangle \quad (\text{symmetric})$$

$$(b) \quad \langle Lx, x \rangle \geq 0 \\ \text{with } \langle Lx, x \rangle = 0 \Leftrightarrow x \equiv 0 \quad (\text{pos def.})$$

Proof of (b):

$$\begin{aligned} \langle Lx, x \rangle &= \int_0^1 - (px')' x + q x^2 dx \\ &= \underbrace{\int_0^1 q(x) x^2 dx}_{\geq 0} - \underbrace{[px' x]_0^1}_{=0} + \int_0^1 p(x) (x')^2 dx \end{aligned}$$

$$\stackrel{(5)}{\geq} \alpha \int_0^1 (x'(x))^2 dx = \alpha \langle x', x' \rangle \geq 0$$

$$\text{And } \langle x', x' \rangle = 0 \ \& \ x(0)=0 \Leftrightarrow x \equiv 0.$$

Consequences:

• If (λ_k, x_k) and (λ_m, x_m) are eigenpairs of (6) with $\lambda_k \neq \lambda_m$ then $\langle x_k, x_m \rangle = 0$ as

$$\lambda_k \langle \underline{x}_k, \underline{x}_m \rangle = \langle L \underline{x}_k, \underline{x}_m \rangle = \langle \underline{x}_m, L \underline{x}_m \rangle = \lambda_m \langle \underline{x}_k, \underline{x}_m \rangle$$

$$\Rightarrow (\lambda_k - \lambda_m) \langle \underline{x}_k, \underline{x}_m \rangle = 0$$

- $\lambda > 0$ in (6), since if $(\lambda, \underline{x})$ is sol, then
 $\lambda \langle \underline{x}, \underline{x} \rangle = \langle L \underline{x}, \underline{x} \rangle > 0$ (since $\underline{x} \neq 0$)
 $\Rightarrow \lambda = \frac{\langle L \underline{x}, \underline{x} \rangle}{\langle \underline{x}, \underline{x} \rangle} > 0.$

Theorem When (5) holds, there exists solutions $(\lambda_k, \underline{x}_k)$ to (6) with
 $0 < \lambda_1 < \lambda_2 < \dots$, $\lim_{k \rightarrow \infty} \lambda_k = \infty$
 and $\underline{x}_k$ orthogonal.

Ex When $p(x) = (1+x)$ and $q(x) = 0$,
 then $-((1+x) \underline{x}')' = \lambda \underline{x}$
 $\underline{x}(0) = \underline{x}(1) = 0$
 has solutions

$$\underline{x}_k(x) = (1+x)^{-\frac{1}{2}} \sin \left[k\pi \frac{\log(1+x)}{\log(2)} \right]$$

$$\lambda_k = \left(\frac{k\pi}{\log(2)} \right)^2 + \frac{1}{4} \quad \bullet \quad (\text{See Exercise 8.15})$$