

Lecture 19

Let $\mathcal{P} = \{f: [a, b] \rightarrow \mathbb{R} \mid f \text{ is piecewise continuous}\}$

and for $f, g \in \mathcal{P}$, define

- $\langle f, g \rangle = \int_a^b f(x) g(x) dx$

- (almost) norm

$$\|f\| := \sqrt{\langle f, f \rangle} = \sqrt{\int_a^b (f(x))^2 dx}$$

and mean square distance

$$\|f - g\| = \sqrt{\int_a^b (f(x) - g(x))^2 dx}$$

It holds that

a) $\|\alpha f\| = |\alpha| \|f\| \quad \forall \alpha \in \mathbb{R} \text{ \& } f \in \mathcal{P}$

b) $\|f + g\| \leq \|f\| + \|g\| \quad \forall f, g \in \mathcal{P} \text{ (will show)}$

c) $\|f\| \geq 0$ and

$$\|f\| = 0 \Leftrightarrow f(x) = 0 \text{ for almost all } x \in [a, b]$$

$\|\cdot\|$ is almost a norm on $\mathcal{P}$

Def 1 A sequence $\{f_N\}_{N=1}^{\infty} \subset \mathcal{P}$

is said to converge in mean square sense (MSS) to $f \in \mathcal{P}$ if

$$\lim_{N \rightarrow \infty} \|f_N - f\| = 0.$$

Example Let $[a, b] = [0, 1]$ and

$f_N(x) = x^N$. Then $f_N \xrightarrow{N \rightarrow \infty} f \equiv 0$ in MSS,

since
$$\|f_N - 0\|^2 = \int_0^1 (f_N(x))^2 dx = \int_0^1 x^{2N} dx = \frac{1}{2N+1} \rightarrow 0 \text{ as } N \rightarrow \infty.$$

NB! Convergence in MSS is not equivalent to pointwise convergence.

For example, $f_N(x) \not\rightarrow 0$ pointwise for all $x \in [0, 1]$, as $f_N(1) \rightarrow 1$.

Lemma 1 (Cauchy - Schwarz ineq)

If $f, g \in \mathcal{P}$, then

$$|\langle f, g \rangle| \leq \|f\| \|g\| \quad (1)$$

Proof If $f(x) = 0$ for almost all $x \in [a, b]$, then (1) becomes $0 \leq 0$, which holds.

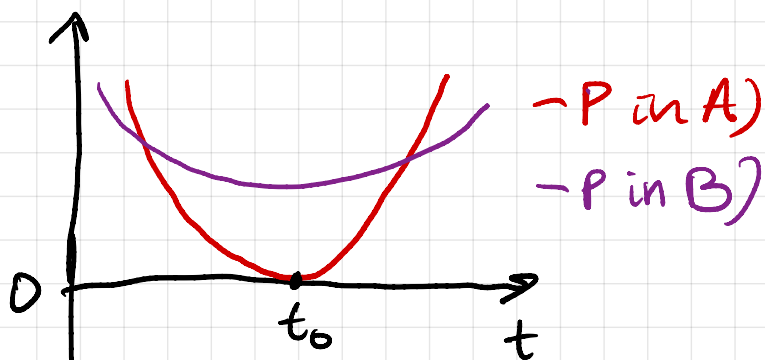
Assume next $\|f\| > 0$ and let

$$\begin{aligned} P(t) &:= \int_a^b (t f(x) - g(x))^2 dx && \text{for } t \in \mathbb{R} \\ &= t^2 \int_a^b f^2(x) dx - 2t \int_a^b f(x)g(x) dx + \int_a^b g^2(x) dx \\ &= t^2 \|f\|^2 - 2t \langle f, g \rangle + \|g\|^2 \end{aligned}$$

Note: $P(t)$ is nonnegative, second order, strictly convex polynomial in t , with roots

$$t_{\pm} = \frac{\langle f, g \rangle \pm \sqrt{\langle f, g \rangle^2 - \|f\|^2 \|g\|^2}}{\|f\|^2} \quad (2)$$

$P \geq 0$ + strictly convex $\Rightarrow$
EITHER A): P has one real double root $t_0 \in \mathbb{R}$ OR B): $P > 0$ and P has complex roots $t_{\pm} \in \mathbb{C}$



Implies in (2) that

$$\langle f, g \rangle^2 - \|f\|^2 \|g\|^2 \leq 0$$

and (1) follows.



Use of Cauchy-Schwarz

Verification of triangle ineq: For $f, g \in \mathcal{D}$

$$\begin{aligned}\|f+g\|^2 &= \langle f+g, f+g \rangle \\ &= \langle f, f \rangle + 2\langle f, g \rangle + \langle g, g \rangle \\ &\stackrel{\text{Cauchy-Schwarz}}{\leq} \|f\|^2 + 2\|f\|\|g\| + \|g\|^2 \\ &= (\|f\| + \|g\|)^2\end{aligned}$$

Taking $\sqrt{\quad}$ of ineq yields

$$\|f+g\| \leq \|f\| + \|g\|.$$

Poincaré inequality

Lemma 2 Let $f \in C([a, b])$ satisfy that f' is piecewise continuous and $f(a) = 0$.

$$\text{Then } \|f\| \leq \frac{(b-a)}{\sqrt{2}} \|f'\|.$$

Proof For any $x \in [a, b]$, $f(x) = \underbrace{f(a)}_{=0} + \int_a^x f'(y) dy$.

$$\begin{aligned}\Rightarrow |f(x)|^2 &= \left(\int_a^x 1 f'(y) dy \right)^2 = \left(\int_a^b \mathbb{1}_{[a, x]}(y) f'(y) dy \right)^2 \\ &= |\langle \mathbb{1}_{[a, x]}, f' \rangle|^2\end{aligned}$$

where $\mathbb{I}_{[a,x]}(y) := \begin{cases} 1 & y \in [a, x] \\ 0 & y \in (x, b] \end{cases}$

By Cauchy-Schwarz

$$|f(x)|^2 \leq \|\mathbb{I}_{[a,x]}\|^2 \|f'\|^2 = (x-a) \|f'\|^2, \text{ as}$$

$$\|\mathbb{I}_{[a,x]}\|^2 = \int_a^b \mathbb{I}_{[a,x]}(y) dy = \int_a^x 1 dy = (x-a).$$

Hence
$$\begin{aligned} \|f\|^2 &= \int_a^b |f(x)|^2 dx \\ &\leq \int_a^b (x-a) dx \cdot \|f'\|^2 \\ &\leq \frac{(b-a)^2}{2} \|f'\|^2 \end{aligned}$$
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Sharpening of energy estimate

Recall that for $u_t = u_{xx} \quad t > 0, x \in (0,1)$
 $u(0,t) = u(1,t) = 0 \quad t \geq 0,$

we showed for $E(t) := \int_0^1 u^2(x,t) dx$

that $E'(t) = -2 \int_0^1 u_x^2(x,t) dx \leq 0, \quad (3)$

and thereby $E(t) \leq E(0) \quad t \geq 0.$

Sharper upper bound $E(t)$:

Since $u(0,t) = 0$ and $u(\cdot, t) \in C^2$,
 Lemma 2 on $[a,b] = [0,1]$ yields

$$E(t) = \int_0^1 u^2(x, t) dx \leq \frac{1}{2} \int_0^1 u_x^2(x, t) dx$$

$$\stackrel{(3)}{=} -\frac{1}{4} E'(t).$$

Hence $E'(t) \leq -4 E(t) \quad | \cdot e^{4t}$

$$\Rightarrow (e^{4t} E(t))' \leq 0$$

$$\Rightarrow E(t) \leq e^{-4t} E(0) \quad t \geq 0$$

and $\lim_{t \rightarrow \infty} E(t) = 0.$

General Fourier Series

Consider $\mathcal{P}$ on interval $[a, b]$

Def For a sequence $\{\mathcal{X}_k\}_{k=1}^{\infty} \subset \mathcal{P}$
 that is orthogonal $\langle \mathcal{X}_k, \mathcal{X}_m \rangle = 0 \quad k \neq m$
 and non-zero $\|\mathcal{X}_k\| \neq 0 \quad \forall k$, and for
 any $f \in \mathcal{P}$, we call

$$S_{\infty}(f) = \sum_{k=1}^{\infty} \frac{\langle f, \mathcal{X}_k \rangle}{\underbrace{\|\mathcal{X}_k\|^2}_{=: c_k}} \mathcal{X}_k$$

the general Fourier series of f .

Let further $S_N(f) := \sum_{k=1}^N c_k \mathcal{X}_k.$

Examples

- Sine series on $[0,1]$ $\Sigma_k(x) = \sin(k\pi x)$ $k \geq 1$
- Example 8.5 TW on $[0,1]$ $\Sigma_k(x) = \sin((k+\frac{1}{2})\pi x)$ $k \geq 0$
- Fourier series on $[-1,1]$: $\Sigma_0 \equiv 1$
$$\left. \begin{aligned} \Sigma_{2k-1}(x) &= \sin(k\pi x) \\ \Sigma_{2k}(x) &= \cos(k\pi x) \end{aligned} \right\} k \geq 1$$

Properties of general Fourier series

For any function on the form $P_N = \sum_{k=1}^N \alpha_k \Sigma_k$, $\alpha_k \in \mathbb{R}$, it holds that

$$\|P_N\|^2 = \sum_{k=1}^N \alpha_k^2 \|\Sigma_k\|^2 \quad (4), \text{ since}$$

$$\begin{aligned} \|P_N\|^2 &= \left\langle \sum_{j=1}^N \alpha_j \Sigma_j, \sum_{k=1}^N \alpha_k \Sigma_k \right\rangle \\ &= \sum_{j=1}^N \sum_{k=1}^N \alpha_j \alpha_k \langle \Sigma_j, \Sigma_k \rangle \quad (\text{orthogonality}) \\ &= \sum_{k=1}^N \alpha_k^2 \|\Sigma_k\|^2. \end{aligned}$$

Lemma 3 Let $f \in \mathcal{P}$. Then for any function on the form $P_N = \sum_{k=1}^N \alpha_k \Sigma_k$ $\alpha_k \in \mathbb{R}$,

it holds that

$$\langle f - S_N(f), P_N \rangle = 0 \quad (\text{orthogonality property})$$

Proof If $1 \leq k \leq N$, then

$$\begin{aligned} \langle S_N(f), \mathcal{X}_k \rangle &= \left\langle \sum_{j=1}^N c_j \mathcal{X}_j, \mathcal{X}_k \right\rangle \\ &= \sum_{j=1}^N c_j \langle \mathcal{X}_j, \mathcal{X}_k \rangle \\ &= c_k \langle \mathcal{X}_k, \mathcal{X}_k \rangle \\ &= \langle f, \mathcal{X}_k \rangle \end{aligned}$$

Hence $\langle f - S_N(f), \mathcal{X}_k \rangle = 0$ for $k=1, 2, \dots, N$
and by linearity

$$\begin{aligned} \langle f - S_N(f), P_N \rangle &= \left\langle f - S_N(f), \sum_{k=1}^N \alpha_k \mathcal{X}_k \right\rangle \\ &= \sum_{k=1}^N \alpha_k \underbrace{\langle f - S_N(f), \mathcal{X}_k \rangle}_{=0} \\ &= 0. \end{aligned}$$



Taking $P_N = S_N(f)$ in Lemma 3
yields $\langle f - S_N(f), S_N(f) \rangle = 0$.

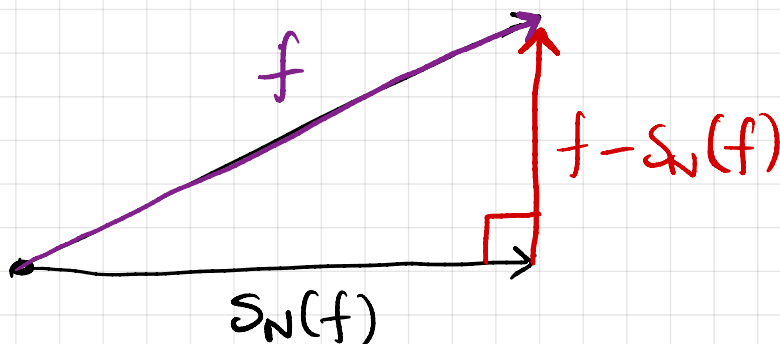
So $(f - S_N(f))$ and $S_N(f)$ are orthogonal,

and

$$\langle f, S_N(f) \rangle = \|S_N(f)\|^2 \stackrel{(4)}{=} \sum_{k=1}^N c_k^2 \|\Sigma_k\|^2 \quad \text{with } P_N = S_N(f)$$

Orthogonal decomposition

$$f = S_N(f) + (f - S_N(f))$$



From

$$\begin{aligned} \|f\|^2 &= \langle S_N(f) + (f - S_N(f)), S_N(f) + (f - S_N(f)) \rangle \\ &= \|S_N(f)\|^2 + 2 \underbrace{\langle f - S_N(f), S_N(f) \rangle}_{=0} + \|f - S_N(f)\|^2, \end{aligned}$$

we obtain Pythagoras theorem

$$\|f\|^2 = \|S_N(f)\|^2 + \|f - S_N(f)\|^2 \quad N \geq 1,$$

and mean square distance (squared)

$$\begin{aligned} 0 \leq \|f - S_N(f)\|^2 &= \|f\|^2 - \|S_N(f)\|^2 \\ &= \|f\|^2 - \sum_{k=1}^N c_k^2 \|\Sigma_k\|^2 \quad (5) \end{aligned}$$

$$(5) \Rightarrow \sum_{k=1}^N c_k^2 \|\Delta_k\|^2 \leq \|f\|^2 \quad \text{for all } N \geq 1$$

and limit $N \rightarrow \infty$ yields

$$\sum_{k=1}^{\infty} c_k^2 \|\Delta_k\|^2 \leq \|f\|^2 \quad (\text{Bessel's inequality})$$

$$\text{Result: } (5) \Rightarrow \lim_{N \rightarrow \infty} \|f - S_N(f)\| = 0$$

if and only if Parseval's identity holds:

$$\sum_{k=1}^{\infty} c_k^2 \|\Delta_k\|^2 = \|f\|^2$$

Examples

Fourier series on $[-1, 1]$: Let $f \in P$ and

$$\begin{aligned} S_{\infty}(f) &= \underbrace{\frac{a_0}{2}}_{c_0} \underbrace{1}_{\Delta_0} + \sum_{k=1}^{\infty} \underbrace{a_k \cos(k\pi x)}_{\underbrace{c_{2k}}_{\Delta_{2k}}} + \underbrace{b_k \sin(k\pi x)}_{\underbrace{c_{2k-1}}_{\Delta_{2k-1}}} \\ &= \sum_{k=0}^{\infty} c_k \Delta_k \end{aligned}$$

$$\|\Delta_0\|^2 = \int_{-1}^1 1^2 dx = 2, \quad \|\Delta_k\|^2 = 1$$

Bessel's ineq:

$$\sum_{k=0}^{\infty} c_k^2 \|\Delta_k\|^2 = 2c_0^2 + \sum_{k=1}^{\infty} c_k^2 \leq \|f\|^2$$

$$\text{or } \frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2) \leq \int_{-1}^1 f^2(x) dx$$

For sine series on $[0,1]$ and $f \in \mathcal{P}$, we have

$$S_{\infty}(f) = \sum_{k=1}^{\infty} c_k \underbrace{\sin(k\pi x)}_{\Delta_k(x)}$$

$$\|\Delta_k\|^2 = \int_0^1 \sin^2(k\pi x) dx = \frac{1}{2}.$$

Bessel's ineq:

$$\sum_{k=1}^{\infty} c_k^2 \|\Delta_k\|^2 \leq \|f\|^2$$

$$\Rightarrow \sum_{k=1}^{\infty} c_k^2 \leq 2 \int_0^1 f^2(x) dx.$$

Application: For $f(x) = x$ on $[0,1]$, one has

$$c_k = 2 \langle f, \sin(k\pi x) \rangle = \frac{2}{\pi} \frac{(-1)^{k+1}}{k}$$

$$\text{and } S_{\infty}(f) = \sum_{k=1}^{\infty} c_k \sin(k\pi x)$$

Bessel's ineq

$$\sum_{k=1}^{\infty} c_k^2 = \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \leq 2 \int_0^1 \underbrace{x^2}_{f^2(x)} dx = \frac{2}{3}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^2} \leq \frac{\pi^2}{6}.$$