

Lecture 20

Let $\mathcal{P}$ = set of piecewise cont. functions on $[a, b]$

For $\{f_N\} \subseteq \mathcal{P}$ and $f \in \mathcal{P}$, recall following types of convergence:

- Mean square sense (MSS) on $[a, b]$ if

$$\lim_{N \rightarrow \infty} \|f_N - f\| = 0, \text{ where } \|f\| = \sqrt{\int_a^b f^2(x) dx}$$

- Uniformly on $[a, b]$ if $\lim_{N \rightarrow \infty} \|f_N - f\|_{\infty} = 0$,

$$\text{where } \|f\|_{\infty} = \sup_{x \in [a, b]} |f(x)|$$

- Pointwise on open/closed set $I \subseteq [a, b]$ if

$$\lim_{N \rightarrow \infty} f_N(x) = f(x) \text{ for any } x \in I.$$

(or equivalently $\lim_{N \rightarrow \infty} |f_N(x) - f(x)| = 0$)

Ex $f_N(x) = e^{-Nx}$ converges in MSS to $f \equiv 0$

on $[0, 1]$:

$$\|f_N - f\|^2 = \int_0^1 (e^{-Nx} - 0)^2 dx = \frac{1 - e^{-2N}}{2N} \rightarrow 0 \text{ as } N \rightarrow \infty.$$

And since

$$\lim_{N \rightarrow \infty} f_N(x) = \lim_{N \rightarrow \infty} e^{-Nx} = \begin{cases} 1 & \text{if } x=0 \\ 0 & \text{if } x \in (0,1] \end{cases},$$

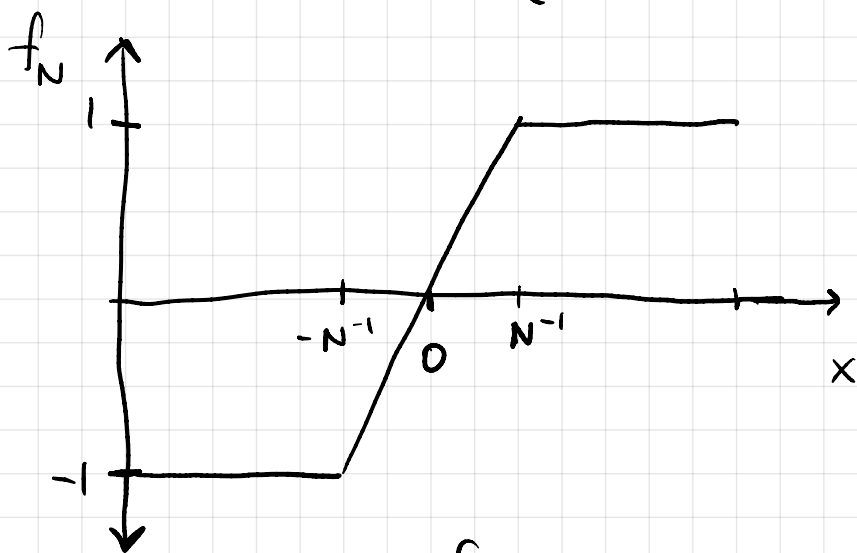
f_N converges pointwise to $\tilde{f}(x) = \begin{cases} 1 & x=0 \\ 0 & x \in (0,1] \end{cases}$

And also uniformly to $\tilde{f}$ on $[0,1]$ as

$$\sup_{x \in [0,1]} |e^{-Nx} - \tilde{f}(x)| = \sup_{x \in [0,1]} |e^{-Nx}| \rightarrow 0 \text{ as } N \rightarrow \infty.$$

Ex

$$\text{Let } f_N(x) = \begin{cases} -1 & x \leq -N^{-1} \\ Nx & x \in (-N^{-1}, N^{-1}) \\ 1 & x \geq N^{-1} \end{cases}$$



$$\text{Let } f(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$$

Then $f_N \xrightarrow{\text{Pointwise}} f$ on $[-1,1]$ since

$f_N(0) = f(0)$, and for any $x \in [-1,1] \setminus \{0\}$

$f_N(x) = f(x)$ when $N \geq \frac{1}{|x|}$.

$$\begin{aligned}\text{And } \|f_N - f\|^2 &= \int_{-N^{-1}}^0 (Nx+1)^2 dx + \int_0^{N^{-1}} (Nx-1)^2 dx \\ &= 2 \int_0^{N^{-1}} (Nx-1)^2 dx = 2N^{-1} \int_0^1 (y-1)^2 dy \\ &\rightarrow 0 \text{ as } N \rightarrow \infty\end{aligned}$$

So $f_N \rightarrow f$ in MSS on $[-1, 1]$

$$\text{But } \|f_N - f\|_\infty = \sup_{x \in [-1, 1]} |f_N(x) - f(x)| = 1$$

for any $N \in \mathbb{N}$, so f_N does not converge uniformly to f .

Properties

Uniform conv $\Rightarrow$ Pointwise conv
and MSS conv

Proof Let $\{f_N\}_{N=1}^\infty \subseteq \mathcal{P}$ satisfy

$$\|f_N - f\|_\infty = \sup_{x \in [a, b]} |f_N(x) - f(x)| \rightarrow 0 \text{ as } N \rightarrow \infty$$

for some $f \in \mathcal{P}$. Then for any $x \in [a, b]$

$$\lim_{N \rightarrow \infty} |f_N(x) - f(x)| \leq \lim_{N \rightarrow \infty} \|f_N - f\|_\infty = 0,$$

and

$$\begin{aligned}\|f_N - f\|^2 &= \int_a^b |f_N(x) - f(x)|^2 dx \leq \|f_N - f\|_\infty^2 (b-a) \\ &\rightarrow 0 \text{ as } N \rightarrow \infty.\end{aligned}$$

Pointwise convergence of Fourier series

$$S_N(f)(x) = \frac{a_0}{2} + \sum_{k=1}^N a_k \cos(k\pi x) + b_k \sin(k\pi x)$$

$$\text{with } a_k = \int_{-1}^1 \cos(k\pi y) f(y) dy$$

$$b_k = \int_{-1}^1 \sin(k\pi y) f(y) dy$$

Alternative representation:

$$S_N(f)(x) = \int_{-1}^1 \left[\frac{1}{2} + \sum_{k=1}^N \underbrace{\cos(k\pi y) \cos(k\pi x) + \sin(k\pi y) \sin(k\pi x)}_{\cos(k\pi(x-y))} \right] f(y) dy$$

$$= \frac{1}{2} \int_{-1}^1 \left[1 + 2 \sum_{k=1}^N \cos(k\pi(x-y)) \right] f(y) dy$$

$$= \frac{1}{2} \int_{-1}^1 K_N(x-y) f(y) dy,$$

$$\text{where } K_N(z) := 1 + 2 \sum_{k=1}^N \cos(k\pi z)$$

is the Dirichlet kernel. Properties:

$$\frac{1}{2} \int_{-1}^1 K_N(z) dz = \frac{1}{2} \int_{-1}^1 1 dz = 1 \quad (1), \text{ and using}$$

$$\begin{aligned} K_N(z) &= 1 + 2 \sum_{k=1}^N \cos(k\pi z) = 1 + \sum_{k=1}^N e^{ik\pi z} + e^{-ik\pi z} \\ &= \sum_{k=-N}^N e^{ik\pi z} = \sum_{k=-N}^N (e^{i\pi z})^k, \end{aligned}$$

one can show that

$$K_N(z) = \begin{cases} 2N+1, & z = 0, \pm 2, \pm 4, \dots \\ \frac{\sin((N+\frac{1}{2})\pi z)}{\sin(\frac{\pi z}{2})}, & \text{otherwise} \end{cases} \quad (2)$$

Let $\mathcal{P}^l$ denote the class of piecewise C^l functions on $[-1, 1]$: $f \in \mathcal{P}^l \Rightarrow$ there exists a finite number of points $-1 = x_0 < x_1 < \dots < x_{k+1} = 1$,
(singular points)

such that $f \in C^l((x_i, x_{i+1}))$ and limits

$$f^{(m)}(x_i^+) = \lim_{x \downarrow x_i} f^{(m)}(x), \quad f^{(m)}(x_{i+1}^-) = \lim_{x \uparrow x_{i+1}} f^{(m)}(x) \text{ exist}$$

for all $i = 0, 1, \dots, k$ and $0 \leq m \leq l$

Ex • $f(x) = |x|$ belongs to $\mathcal{P}^l$ for any $l \geq 0$

$$f \in C^l((-1, 0)) \text{ and } f \in C^l((0, 1))$$

$$\text{with } f(0^\pm) = 0, \quad f'(0^\pm) = \pm 1,$$

$$f^{(m)}(0^\pm) = 0 \text{ for } m \geq 2.$$

• $f(x) = \operatorname{sgn}(x) \in \mathcal{P}^l$ for any $l \geq 0$ with

$$f \in C^l((-1, 0)) \text{ and } f \in C^l((0, 1)), \text{ with}$$

$$f(0^\pm) = \pm 1 \text{ and } f^{(m)}(0^\pm) = 0 \text{ for } m \geq 1.$$

Theorem 1 Let $f \in \mathcal{P}'$ on $[-1, 1)$ and let f_{per} be its 2-periodic extension. Then

$$\lim_{N \rightarrow \infty} S_N(f)(x) = \frac{f_{\text{per}}(x-) + f_{\text{per}}(x+)}{2} \quad \forall x \in [-1, 1]$$

Proof for simplified setting with $f \in \mathcal{P}^2$. Let $x \in [-1, 1]$.

By (1)-(2) and $K_N(z)$ symmetric,

$$S_N(f)(x) - \frac{1}{2}(f(x+) + f(x-))$$

$$= \frac{1}{2} \int_{-1}^1 K_N(x-y) \left[f(y) - \frac{1}{2}(f(x+) + f(x-)) \right] dy$$

$$= \frac{1}{2} \int_{-1-x}^{1-x} K_N(z) \left[f(x+z) - \frac{1}{2}(f(x+) + f(x-)) \right] dz \quad \boxed{z=y-x}$$

$$= \frac{1}{2} \int_{-1}^1 K_N(z) \left[f_{\text{per}}(x+z) - \frac{1}{2}(f(x+) + f(x-)) \right] dz$$

$$= \frac{1}{2} \underbrace{\int_0^1 K_N(z) [f_{\text{per}}(x+z) - f(x+)] dz}_{=: I_N^+} + \frac{1}{2} \underbrace{\int_{-1}^0 K_N(z) [f_{\text{per}}(x+z) - f(x-)] dz}_{=: I_N^-}$$

Let $0 = z_0 < z_1 < \dots < z_{n+1} = 1$ be singular points of $f_{\text{per}}(x+z)$ on $z \in [0, 1]$

$$\text{Then } I_N^+ = \sum_{j=0}^n \int_{z_j}^{z_{j+1}} \underbrace{\sin((N+\frac{1}{2})\pi z)}_{=: g_+(z)} \frac{f_{\text{per}}(x+z) - f(x+)}{\sin(\pi z/2)} dz$$

$$= \frac{1}{(N+\frac{1}{2})\pi} \sum_{j=0}^N \left\{ \left[\cos\left(N+\frac{1}{2}\right)\pi z \right] g_+(z) \right\}_{z_j}^{z_{j+1}} + \int_{z_j}^{z_{j+1}} \cos\left(N+\frac{1}{2}\right)\pi z g'_+(z) dz \Bigg\}$$

$\rightarrow 0$ as $N \rightarrow \infty$,

using that f_{per} is piecewise C^2 so that

$$\sup_{z \in (0,1]} |g_+(z)| < \infty \quad \text{and} \quad \sup_{z \in (0,1]} |g'_+(z)| < \infty.$$

By similar argument, also $I_N^- \rightarrow 0$ $\square$