

Lecture 21

Truncated Fourier series for f on $[-1, 1] =$

$$S_N(f)(x) = \frac{a_0}{2} + \sum_{k=1}^N a_k \cos(k\pi x) + b_k \sin(k\pi x)$$

$$\text{with } a_k = \int_{-1}^1 f(x) \cos(k\pi x) dx$$

$$b_k = \int_{-1}^1 f(x) \sin(k\pi x) dx$$

Theorem 1 Let $f \in C([-1, 1])$ with $f(-1) = f(1)$, and let $f' \in \mathcal{P}$ (Piecewise cont.). Then $S_N(f)$ converges uniformly to f on $[-1, 1]$.

Proof Let f_{per} be 2-periodic extension of f from $[-1, 1)$ to $\mathbb{R}$. Then $f_{\text{per}} \in C(\mathbb{R})$ since $f \in C([-1, 1])$ and $f(-1) = f(1)$.

Since $f \in \mathcal{P}'$, Thm 1 in Lecture 20 implies that for all $x \in [-1, 1]$

$$\lim_{N \rightarrow \infty} S_N(f)(x) = \frac{f_{\text{per}}(x+) + f_{\text{per}}(x-)}{2} \quad \begin{array}{c} \boxed{f \text{ cont.} \ \& \ f(-1) = f(1)} \\ \downarrow \\ \underline{\underline{f(x)}} \end{array}$$

Hence $S_\infty(f)(x) = f(x) \ \forall x$ (pointwise conv), so $f - S_N(f) = S_\infty(f) - S_N(f) = \sum_{k=N+1}^{\infty} a_k \cos(k\pi x) + b_k \sin(k\pi x)$

$$\Rightarrow \|f - S_N(t)\|_\infty = \sup_{x \in [-1,1]} \left| \sum_{k=N+1}^{\infty} a_k \cos(k\pi x) + b_k \sin(k\pi x) \right|$$

$$\leq \sum_{k=N+1}^{\infty} |a_k| + |b_k| \xrightarrow{\text{remains to show}} 0 \text{ as } N \rightarrow \infty$$

$$\text{Let } \alpha_k = \int_{-1}^1 f'(y) \cos(k\pi y) dy = \begin{cases} 0, & k=0 \\ k\pi b_k, & k \geq 1 \end{cases} \quad (1)$$

$$\beta_k = \int_{-1}^1 f'(y) \sin(k\pi y) dy = -k\pi a_k$$

↑ LECTURE 17

Since $f' \in \mathcal{P}$, Bessel's inequality yields

$$\frac{\alpha_0^2}{2} + \sum_{k=1}^{\infty} \alpha_k^2 + \beta_k^2 \leq \int_{-1}^1 (f'(y))^2 dy < \infty$$

By (1),

$$\sum_{k=1}^N |a_k| + |b_k| = \frac{1}{\pi} \sum_{k=1}^N \frac{1}{k} (|\alpha_k| + |\beta_k|)$$

Cauchy-Schwarz

$$\leq \frac{1}{\pi} \left(\underbrace{\sum_{k=1}^N \frac{1}{k^2}}_{\leq \frac{\pi^2}{6}} \right)^{\frac{1}{2}} \left(\underbrace{\sum_{k=1}^N (|\alpha_k| + |\beta_k|)^2}_{\leq 2(\alpha_k^2 + \beta_k^2)} \right)^{\frac{1}{2}}$$

$$\leq \sqrt{\frac{2}{6}} \left(\sum_{k=1}^N \alpha_k^2 + \beta_k^2 \right)^{\frac{1}{2}}$$

$$\leq \frac{\|f'\|}{\sqrt{3}} < \infty \quad \forall N \geq 1$$

$$\Rightarrow \sum_{k=1}^{\infty} |a_k| + |b_k| < \infty$$

$$\Rightarrow \sum_{k=N+1}^{\infty} |a_k| + |b_k| \rightarrow 0 \text{ as } N \rightarrow \infty$$



Example. For $f(x) = |x|$, we have $f \in C([-1, 1])$,

$f' = \text{sgn}(x) \in \mathcal{P}$ and $f(-1) = f(1)$. Theorem 1 applies: $\lim_{N \rightarrow \infty} \|S_N(f) - f\|_\infty = 0$.

• For $f(x) = x^3$, $f \in C([-1, 1])$ and $f' \in \mathcal{P}$, but $f(-1) \neq f(1)$. But [Thm 1, Lecture 20]

(Pointwise conv) applies: $S_\infty(f)(x) = \frac{f_{\text{per}}(x-) + f_{\text{per}}(x+)}{2}$

Particularly:

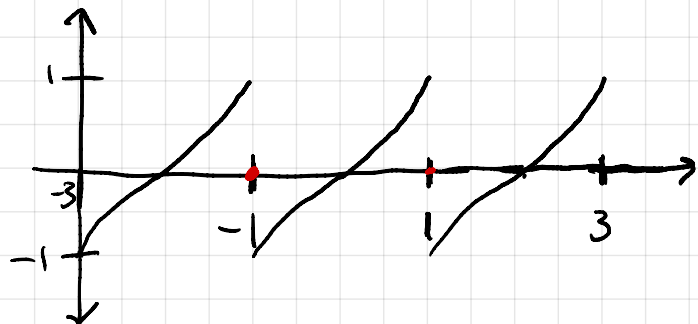
$$S_\infty(f)(-1) = \frac{f_{\text{per}}(-1-) + f_{\text{per}}(-1+)}{2} = \frac{1 + (-1)}{2} = 0$$

$$S_\infty(f)(1)$$

$$S_\infty(f)(3)$$

$$S_\infty(f)(5)$$

:



Uniform convergence does not hold since $S_\infty(f)(-1) \neq f(-1)$ so

$$\lim_{N \rightarrow \infty} \|S_N(f) - f\|_\infty \geq \lim_{N \rightarrow \infty} |S_N(f)(-1) - f(-1)| = 1 \neq 0.$$

Mean square convergence

Recall that $\text{uniform conv} \Rightarrow \text{MSS convergence}$

Means that when Theorem 1 applies,
 conv in MSS also holds.

But conv in MSS holds under more relaxed regularity for f :

Theorem 2

Let $f \in \mathcal{P}$ on $[-1, 1]$. Then
 $S_N(t) \rightarrow f$ in MSS as $N \rightarrow \infty$.

Sketch of proof:

Let $f_{\text{per}}(x)$ be 2-periodic extension of f from $[-1, 1)$ to $\mathbb{R}$, and for any $\delta \in (0, 1)$, let

$$f_\delta(x) := \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} f_{\text{per}}(x) dx \quad x \in \mathbb{R}$$

Properties:

- $\|f - f_\delta\|^2 = \int_{-1}^1 (f(x) - f_\delta(x))^2 dx \rightarrow 0$ as $\delta \rightarrow 0$ (2)

(Exercise 9.17)

- f_δ is 2-periodic

- $f'_\delta(x) = \frac{f_{\text{per}}((x+\delta)-) - f_{\text{per}}((x-\delta)+)}{2\delta}$

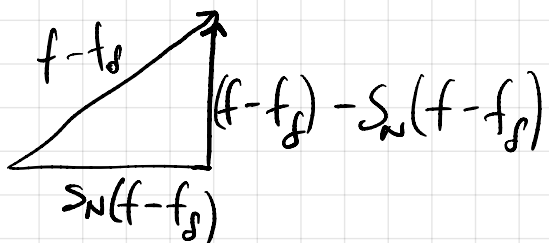
$\Rightarrow f_\delta$ is differentiable and f'_δ is piecewise cont. (since $f_{\text{per}}(x \pm \delta)$ are both piecewise cont).

$\Rightarrow f_\delta \in C([-1, 1])$ (since differentiable) & $f'_\delta \in \mathcal{P}$.

Theorem 1 applies to $f_\delta \Rightarrow$ conv. in MSS:

$$\lim_{N \rightarrow \infty} \|S_N(f_\delta) - f_\delta\| = 0 \quad (3) \quad \text{for any } \delta > 0.$$

By linearity, $S_N(f) - S_N(f_\delta) = S_N(f - f_\delta)$,
and Pythagoras then implies

$$\|S_N(f - f_\delta)\| \leq \|f - f_\delta\|$$


Hence

$$\begin{aligned} \|S_N(f) - f\| &= \|S_N(f - f_\delta) + (S_N(f_\delta) - f_\delta) + (f_\delta - f)\| \\ &\leq 2\|f_\delta - f\| + \|S_N(f_\delta) - f_\delta\| \end{aligned}$$

$$\Rightarrow \lim_{N \rightarrow \infty} \|S_N(f) - f\| \leq 2\|f_\delta - f\| + \underbrace{\lim_{N \rightarrow \infty} \|S_N(f_\delta) - f_\delta\|}_{\substack{(3) \\ = 0}} = 2\|f_\delta - f\|$$

$$\Rightarrow \lim_{N \rightarrow \infty} \|S_N(f) - f\| \leq \lim_{\delta \downarrow 0} 2\|f_\delta - f\| = 0$$



Recall Parseval's identity

$$S_N(t) \rightarrow f \text{ in MSS} \Leftrightarrow \frac{a_0^2}{2} + \sum_{k=1}^{\infty} a_k^2 + b_k^2 = \|f\|^2 (*)$$

Corollary 1 (Thm 2 + Parseval)

If $f \in \mathcal{P}$, then $(*)$ holds

Example Recall that $f(x) = x$ on $[0, 1]$

has sine series

$$S_{\infty}(f)(x) = \sum_{k=1}^{\infty} 2 \underbrace{\frac{(-1)^{k+1}}{k\pi}}_{=b_k} \sin(k\pi x)$$

Odd extension of f on $[-1, 1]$: $\tilde{f}(x) = x$
with Fourier series

$$S_{\infty}(\tilde{f})(x) = \sum_{k=1}^{\infty} 2 \frac{(-1)^{k+1}}{k\pi} \sin(k\pi x)$$

Since $\tilde{f} \in C([-1, 1])$, Corollary 1 $\Rightarrow$

$$\frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} = \|\tilde{f}\|^2 = \int_{-1}^1 x^2 dx = \frac{2}{3}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

Ex For $f(x) = \text{sign}(x)$, $f \in \mathcal{P}$ on $[-1, 1]$,

$$S_{\infty}(f) = \sum_{k=1}^{\infty} \underbrace{\frac{4}{\pi(2k-1)}}_{b_{2k-1}} \sin((2k-1)\pi x)$$

and Corollary 1 $\Rightarrow$

$$\frac{16}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \int_{-1}^1 (\text{sign}(x))^2 dx = 2$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$$

Smoothness of f and decay of $\{a_k\}, \{b_k\}$

By Corollary 1, if $f \in \mathcal{P}$, then

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty} a_k^2 + b_k^2 = \|f\|^2 < \infty.$$

This implies that $\lim_{k \rightarrow \infty} k(a_k^2 + b_k^2) = 0$

In fact, the smoother f is, the faster $\{a_k^2 + b_k^2\}_{k=1}^{\infty}$ will decay.

Let

$$C_p^m = \{f \in C^m([-1, 1]) \mid f^{(j)}(-1) = f^{(j)}(1) \quad j=0, 1, \dots, m\}$$

If $f \in C_p^0$ and $f' \in \mathcal{P}$, then

$$S_{\infty}(f') = \sum_{k=1}^{\infty} \alpha_k (\cos(k\pi x) + \beta_k \sin(k\pi x))$$

with $\alpha_k = k\pi b_k$, $\beta_k = -k\pi a_k$ (Lecture 17)

and Cor 1 $\Rightarrow$

$$\sum_{k=1}^{\infty} \alpha_k^2 + \beta_k^2 = \|f'\|^2 < \infty$$

$$\Rightarrow \sum_{k=1}^{\infty} k^2 (a_k^2 + b_k^2) = \frac{\|f'\|^2}{\pi^2}$$

$$\Rightarrow \lim_{k \rightarrow \infty} k^3 (a_k^2 + b_k^2) = 0.$$

Theorem 3 Let $f \in C_p^{m-1}$ and $f^{(m)} \in \mathcal{P}$ for an integer $m \geq 1$. Then

$$\sum_{k=1}^{\infty} k^{2m} (a_k^2 + b_k^2) = \pi^{-2m} \|f^{(m)}\|^2$$

where a_k, b_k are Fourier coeffs of f .

Proof Result holds for $m=1$ (showed above).

Assume result holds for some $m \geq 1$, and that $f \in C_p^m$ & $f^{(m+1)} \in \mathcal{P}$.

Then $f' \in C_p^{m-1}$ & $\frac{d^m}{dx^m} f' = f^{(m+1)} \in \mathcal{P}$, so it holds (by assumption) that

$$\sum_{k=1}^{\infty} k^{2m} (\alpha_k^2 + \beta_k^2) = \frac{\|f^{(m+1)}\|^2}{\pi^{2m}}$$

where α_k, β_k are Fourier coeffs of f' , so that

$$\alpha_k = \pi k b_k \quad \text{and} \quad \beta_k = -\pi k a_k$$

$$\Rightarrow \sum_{k=1}^{\infty} k^{2(m+1)} (a_k^2 + b_k^2) = \frac{\|f^{(m+1)}\|^2}{\pi^{2(m+1)}}$$

Result holds by induction



Example For $f(x) = \text{sign}(x)$,

$$S_{\infty}(f) = \sum_{k=1}^{\infty} \underbrace{\frac{4}{\pi(2k-1)}}_{b_{2k-1}} \sin((2k-1)\pi x)$$

As $f \in \mathcal{P}$, Cor 1 $\Rightarrow \sum_{k=1}^{\infty} b_k^2 = \|f\|^2 < \infty$,

while $\sum_{k=1}^{\infty} k^2 b_k^2 = \sum_{k=1}^{\infty} \left(\frac{4}{\pi}\right)^2 = \infty$.

Does not contradict Theorem 3 as $f \notin C_p^0$.

Example $f(x) = |x|$ satisfies $f \in C_p^0$ &

$f' = \text{sign}(x) \in \mathcal{P}$, and

$$S_{\infty}(f) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k=1}^{\infty} \left(\frac{1}{2k-1}\right)^2 \cos((2k-1)\pi x).$$

Theorem 3 $\Rightarrow$

$$\sum_{k=1}^{\infty} k^2 a_k^2 = \frac{16}{\pi^4} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\|f'\|^2}{\pi^2} = \frac{2}{\pi^2}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8} \quad (\text{as we know from earlier example}).$$

Will need in next lecture:

Lemma 1 If $\{f_N\}_{N=1}^{\infty} \subset C([a, b])$ and

$$\lim_{N \rightarrow \infty} \|f - f_N\|_{\infty} = \lim_{N \rightarrow \infty} \sup_{x \in [a, b]} |f(x) - f_N(x)| = 0$$

for some $f: [a, b] \rightarrow \mathbb{R}$, then $f \in C([a, b])$

Proof Given $\epsilon > 0$, let $N_0 \in \mathbb{N}$ be such that

$\|f_{N_0} - f\| < \frac{\epsilon}{3}$. Since $f_{N_0} \in C([a, b])$, let $\delta > 0$ be such that

$$|f_{N_0}(x) - f_{N_0}(y)| < \frac{\epsilon}{3} \quad \text{whenever } |x - y| < \delta$$

Then

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_{N_0}(x)| + |f_{N_0}(x) - f_{N_0}(y)| + |f_{N_0}(y) - f(y)| \\ &\leq 2\|f - f_{N_0}\|_{\infty} + \frac{\epsilon}{3} \leq \epsilon \end{aligned}$$

Hence for any $\epsilon > 0$, we can find $\delta > 0$ s.t.

$$|f(x) - f(y)| < \epsilon \quad \text{whenever } |x - y| < \delta.$$

