

Lecture 22

Showed last lecture that

If $f \in C_p^{m-1}$ and $f^{(m)} \in \mathcal{P}$ with Fourier coeffs $\{a_k\}, \{b_k\}$,
then
$$\sum_{k=1}^{\infty} k^{2m} (a_k^2 + b_k^2) = \pi^{-2m} \|f^{(m)}\|^2.$$

Near opposite result:

Theorem 1

If $P_N(x) := \frac{a_0}{2} + \sum_{k=1}^N a_k \cos(k\pi x) + b_k \sin(k\pi x)$

with $\sum_{k=1}^{\infty} k^{2m} (a_k^2 + b_k^2) < \infty$ (1)

then $\exists f \in C_p^{m-1}$ (possibly not with $f^{(m)} \in \mathcal{P}$)
s.t. $P_N^{(j)} \rightarrow f^{(j)}$ for all $0 \leq j \leq m-1$ } (2)
uniformly as $N \rightarrow \infty$. (And $S_{\infty}(t) = P_{\infty}$.)

Proof Two parts: ① for $m=1$

② induction for $m > 1$.

① $m=1$. For each $x \in [-1, 1]$, $\{P_N(x)\}_{N=1}^{\infty}$ is
a Cauchy seq since for any $M > N \geq 1$,

$$|P_M(x) - P_N(x)| \leq \sum_{k=N+1}^M |a_k| + |b_k| = \sum_{k=N+1}^M k^{-1} (k|a_k| + |b_k|)$$

$$\leq \underbrace{\left(\sum_{k=N+1}^M \frac{1}{k^2} \right)^{\frac{1}{2}}}_{\rightarrow 0} \underbrace{\left(\sum_{k=N+1}^M 2k^2(a_k^2 + b_k^2) \right)^{\frac{1}{2}}}_{\rightarrow 0}$$

both as $M, N \rightarrow \infty$.

$\Rightarrow$ pointwise convergence

$$\lim_{N \rightarrow \infty} P_N(x) =: \underbrace{f(x)}_{\text{we call limit } f(x)} \quad \forall x \in [-1, 1]$$

Convergence is also uniform:

$$\begin{aligned} \|f - P_N\|_{\infty} &= \|P_{\infty} - P_N\|_{\infty} \leq \sum_{k=N+1}^{\infty} |a_k| + |b_k| \\ &\stackrel{\text{again}}{\leq} \left(\sum_{k=N+1}^{\infty} \frac{1}{k^2} \right)^{\frac{1}{2}} \left(\sum_{k=N+1}^{\infty} 2k^2(a_k^2 + b_k^2) \right)^{\frac{1}{2}} \\ &\rightarrow 0 \text{ as } N \rightarrow \infty. \end{aligned}$$

Since $P_N \in C_P^0$ for each $N \geq 1$, Lemma 1 last lecture $\Rightarrow f \in C([-1, 1])$, and

$$f(1) = \lim_{N \rightarrow \infty} P_N(1) = \lim_{N \rightarrow \infty} P_N(-1) = f(-1) \Rightarrow f \in C_P^0.$$

Moreover,

Exercise: why?

$$\left. \begin{aligned} \langle f, \cos(k\pi x) \rangle &\stackrel{(*)}{=} \lim_{N \rightarrow \infty} \langle P_N, \cos(k\pi x) \rangle = a_k \\ \langle f, \sin(k\pi x) \rangle &\stackrel{(*)}{=} \lim_{N \rightarrow \infty} \langle P_N, \sin(k\pi x) \rangle = b_k \end{aligned} \right\} \Rightarrow S_{\infty}(f) = P_{\infty}$$

Part ②:

Assume

results $((1) \Rightarrow (2))$ holds for some $m \geq 1$, and
that $\sum_{k=1}^{\infty} k^{2(m+1)} (a_k^2 + b_k^2) < \infty$.

Then $Q_N(x) := P_N'(x) = \sum_{k=1}^{\infty} \alpha_k \cos(k\pi x) + \beta_k \sin(k\pi x)$
with $\alpha_k = k\pi b_k$, $\beta_k = -k\pi a_k$.

Hence

$$\sum_{k=1}^{\infty} k^{2m} (\alpha_k^2 + \beta_k^2) = \pi^2 \sum_{k=1}^{\infty} k^{2(m+1)} (a_k^2 + b_k^2) < \infty,$$

$\Rightarrow$ uniform convergence to $g \in C_p^m$ with

$$Q_N^{(j)} = P_N^{(j+1)} \rightarrow g^{(j)} \quad \text{for } 0 \leq j \leq m-1$$

Know from assumption that for some $f \in C_p^{m-1}$
 $P_N^{(j)} \rightarrow f^{(j)}$ uniformly for $0 \leq j \leq m-1$

$$\Rightarrow g^{(j)} = P_{\infty}^{(j+1)} = f^{(j+1)} \quad \text{for } 0 \leq j \leq m-2.$$

That also $g^{(m-1)} = f^{(m)}$ follows from

Lemma 1 (with $f_N = Q_N^{(m-2)}$ so $\begin{matrix} f_N \rightarrow f^{(m-1)} \\ f_N' \rightarrow g^{(m-1)} \end{matrix}$)

The theorem holds by induction. $\blacksquare$

Lemma 1 Let $\{f_N\} \subset C'([a, b])$ and let there be functions $f, g \in C([a, b])$ s.t.

$\|f - f_N\|_\infty \rightarrow 0$ and $\|g - f'_N\|_\infty \rightarrow 0$ on $[a, b]$ as $N \rightarrow \infty$.

Then $f \in C'([a, b])$ and $f' \equiv g$.

Proof Suffices to show that (by Fund. thm of calculus)
 $f(x) = f(a) + \int_a^x g(y) dy$.

Using that $f_N(x) = f_N(a) + \int_a^x f'_N(y) dy$, we obtain

$$\begin{aligned} |f(x) - f_N(x)| &\leq |f(a) - f_N(a)| + \int_a^x |g(y) - f'_N(y)| dy \\ &\leq \|f - f_N\|_\infty + (x-a) \|g - f'_N\|_\infty \end{aligned}$$

$$\begin{aligned} \Rightarrow \sup_{x \in [a, b]} |f(x) - f_N(x)| &\leq \|f - f_N\|_\infty + (b-a) \|g - f'_N\|_\infty \\ &\rightarrow 0 \text{ as } N \rightarrow \infty. \end{aligned}$$



Convergence of formal solutions

$$\left. \begin{aligned} \text{For } u_t &= u_{xx} & x \in (0,1), t > 0 \\ u(0,t) &= u(1,t) = 0 & t \geq 0 \\ u(x,0) &= f(x), & x \in (0,1) \end{aligned} \right\} (3)$$

with truncated sine series

$$S_N(f)(x) = \sum_{k=1}^N c_k \sin(k\pi x)$$

and truncated formal solution

$$u^N(x,t) = \sum_{k=1}^N c_k e^{-(k\pi)^2 t} \sin(k\pi x)$$

Connection Fourier series on $[-1,1]$

If we seek cont. sol. $u(x,t)$ on $[0,1] \times [0,\infty)$,

then $0 = u(0,0) = \lim_{x \downarrow 0} f(x)$ and $0 = u(1,0) = \lim_{x \uparrow 1} f(x)$

implies that $f(0+) = f(1-)$

$$\text{Let } \tilde{f}(x) = \begin{cases} f(x) & x \in [0,1] \\ -f(-x) & x \in [-1,0) \end{cases} \quad (\text{odd extension})$$

Recall that

$$\underbrace{S_N(\tilde{f})(x)}_{\substack{\text{Fourier} \\ \text{series of } \tilde{f} \\ \text{regu}}} = \underbrace{\sum_{k=1}^N c_k \sin(k\pi x)}_{\text{sine series of } f} \quad \forall N \geq 1$$

Convergence of $S_N(t)$ & u^N depend on regularity of $\tilde{f}$.

For example

$$\tilde{f} \in C_P^0 \text{ (on } [-1, 1]) \Leftrightarrow f \in C([0, 1]) \text{ \& } f(0) = f(1) = 0.$$

Using that $\tilde{f}^{(j)}(-x) = (-1)^{j+1} f^{(j)}(x)$ for $x \in (0, 1]$

one obtains

$$\tilde{f} \in C_P^m \Leftrightarrow \begin{cases} f \in C^m([0, 1]) \text{ \& } \\ f^{(2j)}(0) = f^{(2j)}(1) = 0 \quad 0 \leq 2j \leq m \end{cases}$$

Connection $\tilde{f}$ and f :

Let $C_{P,0}^m = \{ f \in C^m([0, 1]) \mid f^{(2j)}(0) = f^{(2j)}(1) = 0 \quad 0 \leq 2j \leq m \}$,

then $f \in C_{P,0}^m \Leftrightarrow \tilde{f} \in C_P^m$

($C_{P,0}^m$ can be viewed as set of all odd functions in C_P^m .)

Theorem 2

Let $S_\infty(f) = \sum_{k=1}^{\infty} c_k \sin(k\pi x)$ and $m \in \mathbb{N}$,

i) If $f \in C_{p,0}^{m-1}$ and $f^{(m)} \in \mathcal{P}$ on $[0,1]$, then

$$\sum_{k=1}^{\infty} k^{2m} c_k^2 = 2\pi^{-2m} \|f^{(m)}\|^2$$

(Follows from Thm 3, Lecture 21)

ii) If $\sum_{k=1}^{\infty} k^{2m} c_k^2 < \infty$, then $f \in C_{p,0}^{m-1}$
and $S_N^{(j)}(f) \xrightarrow{\text{uniformly}} f^{(j)}$ for $0 \leq j \leq m-1$
(From Thm 1, this lecture)

Connection to $u^N(\cdot, t) = \sum_{k=1}^N \underbrace{c_k e^{-(k\pi)^2 t}}_{=: \bar{c}_k(t)} \sin(k\pi x) =$

i) if $u^\infty(\cdot, t) \in C_{p,0}^{m-1}$ and $(\partial_x^m u^\infty(\cdot, t)) \in \mathcal{P}$,

then $\sum_{k=1}^{\infty} k^{2m} (\bar{c}_k(t))^2 < \infty$

ii) if $\sum_{k=1}^{\infty} k^{2m} (\bar{c}_k(t))^2 < \infty$, then

$u^\infty(\cdot, t) \in C_{p,0}^{m-1}$ and

$\frac{\partial^j}{\partial x^j} u^N(\cdot, t) \xrightarrow{\text{uniformly}} \frac{\partial^j}{\partial x^j} u^\infty(\cdot, t)$ for $0 \leq j \leq m-1$,

(where $\frac{\partial^j}{\partial x^j} u^\infty(x, t) = \sum_{k=1}^{\infty} \bar{c}_k(t) \frac{\partial^j}{\partial x^j} \sin(k\pi x)$.)