

Lecture 23

Heat eq.

$$\left. \begin{aligned} u_t &= u_{xx} & x \in (0,1), t > 0 \\ u(0,t) &= u(1,t) & t \geq 0 \\ u(x,0) &= f(x) & x \in (0,1) \end{aligned} \right\} (1)$$

Truncated formal sol

$$u^N(x,t) = \sum_{k=1}^N \underbrace{C_k e^{-(k\pi)^2 t}}_{=: \bar{C}_k(t)} \sin(k\pi x) \quad (2)$$

$$C_k = \bar{C}_k(0) = 2 \int_0^1 f(y) \sin(k\pi y) dy.$$

(Will show: if $f \in P$, then u^∞ solves (1).)

Recall: $C_{P,0}^m = \{f \in C^m([0,1]) \mid f^{(2j)}(0) = f^{(2j)}(1) = 0, 0 \leq 2j \leq m\}$,

where $f \in C_{P,0}^m \Leftrightarrow \underbrace{\tilde{f}}_{\text{odd extension}} \in C_P^m$ on $[-1,1]$.

Theorem 1

For fixed $t \geq 0$ and $\{C_k(t)\}_{k=1}^\infty$ in (2),
if $\sum_{k=1}^\infty k^{2m} (\bar{C}_k(t))^2 < \infty$, then

$u^\infty(\cdot, t) \in C_{P,0}^{m-1}$ and

$\frac{\partial^j}{\partial x^j} u^N(\cdot, t) \xrightarrow{\text{uniformly}} \frac{\partial^j}{\partial x^j} u^\infty(\cdot, t)$ for $0 \leq j \leq m-1$,

Where $\frac{\partial^j}{\partial x^j} u^\infty(x, t) = \sum_{k=1}^{\infty} \bar{C}_k(t) \frac{\partial^j}{\partial x^j} \sin(k\pi x)$.

To bound $\{k^{2m} e^{-(k\pi)^2 t}\}$ later, we need:

Lemma 1 Let $a, b > 0$. Then there exists a constant $M > 0$, depending on a, b , such that

$$0 \leq x^a e^{-bx} \leq M \quad \forall x \geq 0.$$

Proof Let $g(x) = x^a e^{-bx}$

Then $g(0) = 0$, $g(x) > 0 \quad \forall x \in (0, \infty)$, and $\lim_{x \rightarrow \infty} g(x) = 0$.

$$g'(x) = x^{a-1} e^{-bx} (a - bx)$$

$$\Rightarrow \max_{x \in \mathbb{R}} g(x) = g\left(\frac{a}{b}\right) = \left(\frac{a}{b}\right)^a e^{-a} =: M.$$



Theorem 2 If $f \in \mathcal{P}$ on $[0, 1]$, then for any $t > 0$, $u^\infty(\cdot, t) \in C_{P,0}^\infty$.

Proof For any $m \in \mathbb{N}$ and $t > 0$,

$$\sum_{k=1}^{\infty} k^{2m} (\bar{C}_k(t))^2 = \sum_{k=1}^{\infty} \underbrace{(k^2)^m e^{-k^2 \pi^2 t}}_{\leq M} C_k^2$$

$$\leq M \sum_{k=1}^{\infty} C_k^2 \leq 2M \|f\|^2 < \infty$$

$$(k^2)^m e^{-\pi^2 t k^2}$$

$\begin{matrix} a & b & x \\ \swarrow & \downarrow & \searrow \\ (k^2)^m & e^{-\pi^2 t k^2} & k^2 \end{matrix}$

Lemma 1

Theorem 1 implies that

$$u^\infty(\cdot, t) \in C_{p,0}^m \quad \forall m \geq 1 \text{ and}$$

$$\frac{\partial^j}{\partial x^j} u^N(\cdot, t) \xrightarrow{\text{uniformly}} \frac{\partial^j}{\partial x^j} u^\infty(\cdot, t) \quad \forall j \geq 0.$$

(so $u(\cdot, t)$ much smoother than f !) $\square$

In particular

$$\begin{aligned} u_{xx}^\infty(x, t) &= \sum_{k=1}^{\infty} c_k e^{-(k\pi)^2 t} \frac{\partial^2}{\partial x^2} \sin(k\pi x) \\ &= - \sum_{k=1}^{\infty} c_k e^{-(k\pi)^2 t} (k\pi)^2 \sin(k\pi x) \quad (4) \end{aligned}$$

Theorem 3 If $f \in \mathcal{P}$ on $[0, 1]$, then

$u^\infty(x, t)$ satisfies

$$u_t^\infty = u_{xx}^\infty \quad x \in (0, 1), \quad t > 0$$

$$u^\infty(0, t) = u^\infty(1, t) = 0 \quad t > 0$$

Proof B(are) fulfilled as $u^\infty(\cdot, t) \in C_{p,0}^\infty$.

$$\begin{aligned} \text{And} \quad u_t^N(x, t) &= \sum_{k=1}^N c_k \left(\frac{\partial}{\partial t} e^{-(k\pi)^2 t} \right) \sin(k\pi x) \\ &= - \sum_{k=1}^N c_k (k\pi)^2 \sin(k\pi x) \\ &= u_{xx}^N(x, t) \quad \forall (x, t) \in [0, 1] \times (0, \infty). \end{aligned}$$

Fix $x \in [0, 1]$ and time interval $[\delta, T] \subset (0, \infty)$,
for some $T > \delta > 0$.

Then $\forall t \in [\delta, T]$,

$$|u_t^N(x, t) - u_{xx}^\infty(x, t)| = \pi^2 \left| \sum_{k=N+1}^{\infty} k^2 e^{-(k\pi)^2 t} c_k \sin(k\pi x) \right|$$

$$\leq \pi^2 \sum_{k=N+1}^{\infty} \underbrace{k^3 e^{-k^2 \pi^2 \delta}}_{\substack{\text{Lemma 1} \\ \leq M}} \frac{|c_k|}{k}$$

$$\leq M \pi^2 \sum_{k=N+1}^{\infty} \frac{|c_k|}{k}$$

$$\leq M \pi^2 \underbrace{\left(\sum_{k=N+1}^{\infty} k^{-2} \right)^{\frac{1}{2}}}_{\rightarrow 0} \underbrace{\left(\sum_{k=N+1}^{\infty} c_k^2 \right)^{\frac{1}{2}}}_{\rightarrow 0}$$

uniformly on $[\delta, T]$ as $N \rightarrow \infty$.

Similarly,

$$|u^N(x, t) - u^\infty(x, t)| \leq M \underbrace{\left(\sum_{k=N+1}^{\infty} k^{-2} \right)^{\frac{1}{2}}}_{\rightarrow 0} \underbrace{\left(\sum_{k=N+1}^{\infty} c_k^2 \right)^{\frac{1}{2}}}_{\rightarrow 0}$$

uniformly on $[\delta, T]$ as $N \rightarrow \infty$.

So for any fixed $x \in [0, 1]$, the sequence of fns
 $g_N(t) := u^N(x, t)$ on $[\delta, T]$

converges $g_N \rightarrow g = u^\infty(x, \cdot)$ uniformly
 and $u_t^\infty(x, \cdot) = g_N' \rightarrow u_{xx}^\infty(x, \cdot)$ —||—

By Lemma 1, Lecture 22, since also $g_N, g_N' \in C([\delta, T])$
 for all $N \geq 1$, we $u_t^\infty(x, t) = g'(t) = u_{xx}^\infty(x, t) \quad \forall t \in [\delta, T]$

Argument holds for any $T > \delta > 0$ and $x \in [0, 1]$

$$\Rightarrow u_t^\infty = u_{xx}^\infty \quad \text{on } (0, 1) \times (0, \infty)$$

□

Initial condition

Does $u^\infty(\cdot, t) \rightarrow f$ as $t \downarrow 0$? (in what sense?)

Depends on regularity of f .

If f is piecewise cont but not cont, then $u^\infty(\cdot, t)$ cannot converge uniformly to f as $t \downarrow 0$, since

$\lim_{t \downarrow 0} u^\infty(\cdot, t) \in C([0, 1])$ while $f \notin C([0, 1])$.

(For low-regularity f , we have MSS conv:)

Theorem 4

If $f \in \mathcal{P}$, then $u^\infty(\cdot, t) \rightarrow f$ in MSS on $(0, 1)$ as $t \downarrow 0$.

Proof $f \in \mathcal{P} \Rightarrow S_N(f) = \sum_{k=1}^N c_k \sin(k\pi x)$

conv. in MSS to $f : \|S_N(f) - f\| \rightarrow 0$ as $N \rightarrow \infty$

(as odd extension $\tilde{f} \in \mathcal{P}$ on $[-1,1]$ and $S_N(\tilde{f}) \xrightarrow{\text{MSS}} \tilde{f}$ on $[-1,1]$)

Let $\lfloor t^{-1/3} \rfloor = \text{greatest integer} \leq t^{-1/3}$.

Then for $t \in (0,1)$,

$$\|u^\infty(\cdot, t) - f\|^2 = \|u^\infty(\cdot, t) - S_\infty(f)\|^2$$

$$= \frac{1}{2} \sum_{k=1}^{\infty} (1 - e^{-(k\pi)^2 t}) c_k^2$$

$$\leq \frac{1}{2} \sum_{k=1}^{\lfloor t^{-1/3} \rfloor} \underbrace{(1 - e^{-(k\pi)^2 t})}_{\leq (1 - e^{-\pi^2 t^{1/3}})} c_k^2 + \frac{1}{2} \sum_{k=\lfloor t^{-1/3} \rfloor + 1}^{\infty} c_k^2$$

$$\leq \underbrace{(1 - e^{-\pi^2 t^{1/3}})}_{\rightarrow 0} \underbrace{\sum_{k=1}^{\infty} c_k^2}_{\|f\|^2} + \frac{1}{2} \underbrace{\sum_{k=\lfloor t^{-1/3} \rfloor + 1}^{\infty} c_k^2}_{\rightarrow 0}$$

both as $t \rightarrow 0$



(More regularity on f implies stronger convergence $u^\infty(\cdot, t) \rightarrow f$:)

Theorem 5

If $f \in C_{p,0}^0$ and $f' \in \mathcal{P}$ on $[0,1]$, then $u^\infty(\cdot, t) \rightarrow f$ uniformly as $t \downarrow 0$.

Proof Odd extension $\tilde{f}$ on $[-1,1]$ satisfies $\tilde{f} \in C_p^0$ and $\tilde{f}' \in \mathcal{P}$, which implies that

$$\lim_{N \rightarrow \infty} \|S_N(\tilde{f}) - \tilde{f}\|_\infty = 0 \quad \text{on } [-1,1]$$

And since $S_N(f) = S_N(\tilde{f})$, we also have

$$\lim_{N \rightarrow \infty} \|S_N(f) - f\|_\infty = 0 \quad \text{on } [0,1] \quad (5)$$

Notation, as before, $u^N = \sum_{k=1}^N c_k e^{-(k\pi)^2 t} \sin(k\pi x)$

$$S_N(f) = \sum_{k=1}^N c_k \sin(k\pi x)$$

For any fixed N ,

$$\begin{aligned} \|u^N(\cdot, t) - S_N f\|_\infty &\leq \sum_{k=1}^N (1 - e^{-(k\pi)^2 t}) |c_k| \\ &\leq (1 - e^{-(N\pi)^2 t}) \sum_{k=1}^N |c_k| \rightarrow 0 \text{ as } t \downarrow 0 \end{aligned}$$

And for any fixed $t > 0$ and $M \geq N$,

$$\begin{aligned} \|u^N(\cdot, t) - u^M(\cdot, t)\|_\infty &= \sup_{x \in [0, 1]} \left| \sum_{k=N+1}^M e^{-(k\pi)^2 t} c_k \sin(k\pi x) \right| \\ &\leq \sup_{x \in [0, 1]} \left| \sum_{k=N+1}^M c_k \sin(k\pi x) \right| \\ &= \|S_N(t) - S_M(t)\|_\infty \end{aligned}$$

$\Rightarrow$

$$\lim_{M \rightarrow \infty} \|u^N(\cdot, t) - u^M(\cdot, t)\|_\infty \leq \lim_{M \rightarrow \infty} \|S_N(t) - S_M(t)\|_\infty$$

|| Theorem 1 || (5)

$$\|u^N(\cdot, t) - u^\infty(\cdot, t)\|_\infty \leq \|S_N(t) - f\|_\infty$$

Using this,

$$\begin{aligned} \|u^\infty(\cdot, t) - f\|_\infty &\leq \|u^\infty(\cdot, t) - u^N(\cdot, t)\|_\infty + \|u^N(\cdot, t) - S_N(t)\|_\infty \\ &\quad + \|S_N(t) - f\|_\infty \\ &\leq \|u^N(\cdot, t) - S_N(t)\|_\infty + 2\|S_N(t) - f\|_\infty \end{aligned}$$

Taking $\lim_{N \rightarrow \infty} \lim_{t \downarrow 0}$ on both sides yields

$$\begin{aligned} \lim_{t \downarrow 0} \|u^\infty(\cdot, t) - f\|_\infty &\leq \underbrace{\lim_{N \rightarrow \infty} \lim_{t \downarrow 0} \|u^N(\cdot, t) - S_N(t)\|_\infty}_{=0} \\ &\quad + \underbrace{\lim_{N \rightarrow \infty} 2\|S_N(t) - f\|_\infty}_{=0} \\ &= 0 \end{aligned}$$



Consequence: If $f \in C_{p,0}^0$, then solution $u^\infty \in C([0,1] \times [0,T])$ & $u_t^\infty, u_x^\infty, u_{xx}^\infty \in C((0,1) \times (0,T])$ for any $T > 0 \Rightarrow$ Max. principle Cor 6.1 in TW applies: $\|u(\cdot, t)\|_\infty \leq \|f\|_\infty \quad t \geq 0$.

If f has more regularity, then so does the solution u^∞ :

Theorem 6

If f in PDE (1) satisfies $f \in C_{p,0}^{2j}$ & $f^{(2j+1)} \in \mathcal{P}$ for some $j \geq 0$, then

$$\left\| \frac{\partial^j}{\partial t^j} u^\infty(\cdot, t) \right\|_\infty = \left\| \frac{\partial^{2j}}{\partial x^{2j}} u^\infty(\cdot, t) \right\|_\infty \leq \|f^{(2j)}\|_\infty \quad t \geq 0 \quad (6)$$

Sketch of proof

$j = 0$ ok by above argument.

For $j \geq 1$, extension of Thm 3 yields:

- $\frac{\partial^k}{\partial t^k} u^\infty(x, t) = \frac{\partial^{2k}}{\partial x^{2k}} u^\infty(x, t)$ for any $k \in \mathbb{N}$ on $(0,1) \times (0, \infty)$

- $\frac{\partial^{2j}}{\partial x^{2j}} u^\infty(x, 0) = f^{(2j)}(x) \quad \forall x \in [0,1]$

- And $v(x, t) := \frac{\partial^{2j}}{\partial x^{2j}} u(x, t)$ solves heat eq.

$$V_t = V_{xx} \quad \text{on } (0,1) \times (0,\infty)$$

$$V(0,t) = V(1,t) = 0 \quad t \geq 0$$

$$V(x,0) = f^{(2j)}(x) \quad x \in (0,1)$$

Lastly, Max principle applied to v
proves (6). ■