

Summary Mat3360 UiO

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- 1.1-1.4 Terminology and method of characteristics
- 2.1-2.4 (excluding 2.2.3). Poisson's eq 1D, numerical solution, properties of Laplace operators
- 3.1-3.8 Heat eq. Particular solutions, formal solutions, energy arguments.
- 4.1-4.5 Finite difference methods, von Neumann stability analysis, energy arguments for numerical methods.
- 5.1-5.3 Wave eq. Particular solutions, formal solutions, energy arguments, finite difference method, stability analysis.
- 6.1-6.4 Maximum principles for Poisson's eq in 1D and 2D, linear heat equation, and for numerical solution. Harmonic functions.

- 7.1-7.5 Poisson's equation in 2D. Divergence theorem, polar coordinates.
- 8.1-8.5 Fourier series on $[-1, 1]$, odd and even functions, differentiation of Fourier series.
- 9.1-9.5 Different notions of convergence for sequences of functions on $[a, b]$. Convergence of Fourier series. Relation between smoothness of f and decay of Fourier coefficients of f .
- 10.1-10.2 Formal solution of (the standard) heat equation is indeed the unique exact solution when f is sufficiently smooth.

- Terminology for differential equations (linear vs nonlinear, differential operator of homogeneous part, PDE vs ODE, order, homogeneous vs nonhomogeneous). Example:

$$u_t + u_{xtt} = -e^u + 2 \quad \text{has homogeneous part} \quad L(u) = u_t + u_{xtt} + e^u = 0$$

Is nonlinear, homogeneous part has differential operator $L(u)$, is a PDE, is third order, and is nonhomogeneous.

- Method of characteristics for first-order linear PDE:

$$u_t + a(x, t)u_x = g(x, t) \quad x \in \mathbb{R}, \quad t > 0,$$

given initial condition $u(x, 0)$ for $x \in \mathbb{R}$ and for reasonable functions a and g .

- Poisson's equation

$$-u'' = f \quad x \in (0, 1)$$

with Dirichlet 0-BC. Find explicit solution by integration Uniqueness.

- Bound of solution in terms of $\|f\|_\infty$ using e.g. a Green's function/maximum principle (see Chap 6).
- Finite difference method for solving Poisson's equation. Existence and uniqueness of discrete solution v and maximum principle (return in Chap 6).
- Properties of the operator $Lu = -u''$ on C_0^2 and discrete operator L_h on $D_{h,0}$.
- Truncation error $\tau_h(x_i) = L_h(u)(x_i) - f(x_i) = \mathcal{O}(h^2)$, and know how this error bound can be utilized to show the convergence rate $\|u - v\|_{h,\infty} = \mathcal{O}(h^2)$.
- Know how to solve the Poisson equation with other BC.

- Particular solutions of the heat equation

$$u_t = u_{xx} \quad (x, t) \in (0, 1) \times (0, \infty)$$

with Dirichlet-0 boundary conditions and IC $u(\cdot, 0) = f$ given, through using separation of variables.

- Formal solution by principle of superposition.
- Energy arguments for heat equation: study evolution of

$$E(t) = \int_0^1 u^2(x, t) dx$$

and exploit to obtain uniqueness if $E'(t) \leq 0$.

- Extension to other BC, additional linear terms in PDE of form we have considered in exercises.

- Explicit and implicit finite difference schemes linear PDE with constant coefficients

$$u_t = \alpha u_{xx} + \beta u + \gamma u_x \quad x \in (0, 1), \quad t > 0$$

given Dirichlet or Neumann BC and where $\alpha > 0$, $\beta, \gamma \in \mathbb{R}$.

- Stability analysis for numerical scheme in terms of discrete energy:
Impose conditions on Δt and Δx such that

$$E^m = \Delta x \sum_{j=1}^n (v_j^m)^2$$

is non-increasing in m . Conclude that solution is unique (and well-behaved in the sense that it does not explode).

- von Neumann stability analysis for above PDE with Dirichlet or Neumann BC 0 BC:

- 1 Insert discrete-problem particular solution $v_j^m = (a_k)^m e^{ik\pi x_j}$ for $k \in \mathbb{Z}$ into numerical scheme solving heat equation, example (explicit scheme)

$$\frac{v_j^{m+1} - v_j^m}{\Delta t} = \frac{v_{j-1}^m - 2v_j^m + v_{j+1}^m}{\Delta x^2}$$

- 2 Solve for a_k (this example: $a_k = 1 - 4(\Delta t / \Delta x^2) \sin(k\pi \Delta x / 2)$).
- 3 Test if $|a_k|^m \leq |T_k(t_m)|$ for all $k \in \mathbb{Z}$ and $m \geq 0$, where $T_k(t)e^{ik\pi x}$ are associated continuous-problem particular solutions.
- 4 If true, then scheme is von Neumann stable.

- Separation of variables for wave equation

$$\begin{aligned}u_{tt} &= u_{xx} & x \in (0, 1), t > 0 \\ u(x, 0) &= f(x), \quad u_t(x, 0) = g(x) & x \in (0, 1)\end{aligned}$$

with Dirichlet or Neumann 0 BC.

- Formal solutions adapting Fourier coefficients to initial conditions.
- Energy argument for $E(t) = \int_0^1 u_x^2(x, t) + u_t^2(x, t) dx$ and uniqueness of smooth solutions.
- Explicit finite difference method for solving PDE, uniqueness of solution and von Neumann stability analysis.
- Extension to PDE covered in exercises, for example

$$u_{tt} = u_{xx} + \alpha u_t$$

and/or Dirichlet non-zero BC.

Maximum principles for:

- Maximum principles for harmonic functions

$$\Delta u = 0 \quad \text{in } \Omega \subset \mathbb{R}^d, \quad u = g \quad \text{on } \partial\Omega$$

for $d = 1$ and $d = 2$. Know how to show that

$$\min_{\partial\Omega} g \leq u(x) \leq \max_{\partial\Omega} g \quad \forall x \in \Omega.$$

- Exploit to show uniqueness of smooth solutions of Poisson's equation

$$-\Delta u = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial\Omega$$

- Subharmonicity: if u satisfies

$$\Delta u \geq 0 \quad \text{on } \Omega \quad u = g \quad \text{on } \partial\Omega$$

then $u(x) \leq \max_{\partial\Omega} g$ for all $x \in \Omega$.

- For $\Omega = [0, 1]$ or $\Omega = [0, 1]^2$ in 2D, the PDE

$$\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

satisfies maximum principle

$$u \leq \frac{\|f\|_{\infty}}{8}$$

Can be obtained by utilizing

$$w(x) = \|f\|_{\infty} \frac{x(1-x)}{2} \quad \text{in 1D,} \quad w(x, y) = \|f\|_{\infty} \frac{x(1-x)}{2} \quad \text{in 2D.}$$

It solves PDE

$$-\Delta w = \|f\|_{\infty} \quad \text{in } \Omega.$$

and $u - w$ is subharmonic: $\Delta(u - w) \geq 0$ on Ω .

- Then maximum principle on $u - w$ gives $u \leq \|f\|_{\infty}/8$.

- Lower bound $u \geq -\|f\|_\infty/8$ can be obtained by similar argument by exploiting that $(u + w)$ is superharmonic.
- For 1D Poisson's eq. this is an alternative to using the Green's function solution representation to obtain maximum principle (see Chapter 2).
- Discrete equation

$$L_h v = 0 \quad \text{on } \Omega_h, \quad v = g \quad \text{on } \partial\Omega_h$$

on grid Ω_h on line $[0, 1]$ for $d = 1$ or box $[0, 1]^2$ when $d = 2$ satisfies

$$\min_{\partial\Omega_h} g \leq v \leq \max_{\partial\Omega_h} g$$

This is a consequence of

$$v_j = \frac{v_{j-1} + v_{j+1}}{2} \quad \text{in 1D,} \quad v_{jk} = \frac{v_{j-1,k} + v_{j+1,k} + v_{j,k-1} + v_{j,k+1}}{4} \quad \text{in 2D.}$$

- Similar as for the continuous problem, any function satisfying

$$L_h v \leq 0 \quad \text{on } \Omega_h, \quad v = g \quad \text{on } \partial\Omega_h$$

is called discrete subharmonic and satisfies

$$v \leq \max_{\partial\Omega_h} g.$$

- And for

$$L_h v = f \quad \text{in } \Omega_h, \quad v = 0 \quad \text{on } \partial\Omega_h,$$

it holds that

$$v \leq \frac{\|f\|_{h,\infty}}{8}$$

Can be shown by introducing $w = \|f\|_{\infty} x(1-x)/2$ and apply discrete max principle to $L_h(v-w) \leq 0$. (Similar lower bound is obtained using the discrete superharmonic function $u+w$).

- Maximum principle for heat equation for $(x, t) \in [0, 1] \times [0, T] = R$ with Dirichlet BC.

$$\begin{aligned} u_t &= u_{xx} & x \in (0, 1), t \in (0, T] \\ u(0, t) &= u_l(t), \quad u(1, t) = u_r(t), & t \in [0, T] \\ u(x, 0) &= f(x) & x \in [0, 1] \end{aligned}$$

Then for $B = \partial R \setminus ([0, 1] \times \{T\})$, lower boundary of R , smooth solutions of PDE satisfy

$$\inf_B u \leq u(x, t) \leq \sup_B u \quad \forall (x, t) \in R.$$

- Extension to other heat-equation type PDE, as covered in exercises.

- By induction argument, one obtains discrete maximum principle for numerical solutions of heat equation (provided condition for Δt and Δx holds).

$$\inf_{B_\Delta} v \leq v(x, t) \leq \sup_{B_\Delta} v \quad \forall (x, t) \in R_\Delta.$$

Topics on Poisson's equation in 2D:

$$-\Delta u = f \quad \text{on } \Omega, \quad u = g \quad \text{on } \partial\Omega.$$

- Formal solution on rectangular domains.
- Formulate PDE in polar coordinates (if question on exam, formulation will be given), for obtaining formal solutions on discs, wedges, annuli, and exteriors of discs.
- Divergence theorem, for $f : \Omega \rightarrow \mathbb{R}^2$ given by $f(x, y) = (f_1(x, y), f_2(x, y))$, it holds under sufficient regularity that

$$\iint_{\Omega} \nabla \cdot f(x, y) dx dy = \int_{\partial\Omega} f \cdot n ds.$$

- Application: For operator $L(u) = -\Delta u = -\nabla \cdot \nabla u$ on $C_0^2(\Omega)$, we see that for $v \in C_0^2$, the function $f = v \nabla u$ is zero on $\partial\Omega$, and

$$\nabla \cdot f = \nabla v \cdot \nabla u + v \Delta u \implies v \Delta u = \nabla \cdot f - \nabla v \cdot \nabla u$$

So divergence theorem/integration by parts yields

$$\begin{aligned} \langle L(u), v \rangle &= \iint_{\Omega} -v \Delta u \, dx dy \\ &= \iint_{\Omega} \nabla v \cdot \nabla u - \nabla \cdot f \, dx dy \\ &= \iint_{\Omega} \nabla v \cdot \nabla u \, dx dy \end{aligned}$$

This can be used to show that L is positive definite and symmetric on C_0^2 .

- These properties, symmetry and positive definiteness, extend to any operator

$$L(u) = -\nabla \cdot (a(x, y) \nabla u)$$

with smooth and scalar-valued function $a \geq \alpha > 0$ on $\overline{\Omega}$.

- Linearity and positive definiteness of L can be used to show uniqueness of smooth solutions of Poisson's equation

$$Lu = f \quad \text{on } \Omega, \quad u = g \text{ on } \partial\Omega.$$

- Mean value property for harmonic functions, if $\Delta u = 0$ on Ω then for any $(x, y) \in \Omega$ and $a > 0$ s.t. $B_a((x, y)) \subset \Omega$,

$$u(x, y) = \frac{1}{2\pi a} \int_{\partial B_a((x, y))} u \, ds$$

- Given $\Omega = [0, 1]^2$ and Ω_h grid with stepsize h in both directions on Ω , know numerical method with five-point operator for solving discrete Poisson equation $L_h v = f$ with $v \in D_{h,0}$.
- Show that L_h is positive definite and symmetric on $D_{h,0}$ wrt inner product $\langle \cdot, \cdot \rangle_h$ and conclude that there exists a unique numerical solution to problem in previous bullet.
- For discrete solution $L_h v = f$ and continuous solution $Lu = f$, both with Dirichlet 0 BC, consider truncation error

$$\tau_h(x, y) = L_h u(x, y) - f(x, y) = \mathcal{O}(h^2) \quad \forall (x, y) \in \Omega_h$$

Then $e = u - v$ solves

$$L_h e = \tau_h$$

with $e = 0$ on $\partial\Omega_h$ and discrete max principle (Chp 6) yields

$$\|e\|_{h,\infty} = \mathcal{O}(\|\tau_h\|_{h,\infty}) = \mathcal{O}(h^2).$$

- Recall that $\mathcal{P}[a, b]$ denotes the space of piecewise continuous functions on $[a, b]$ which we equip with the inner product

$$\langle f, g \rangle = \int_a^b f(x)g(x) dx,$$

and introduce the mean square distance norm on $\mathcal{P}[a, b]$

$$\|f\| := \sqrt{\langle f, f \rangle}.$$

- Result: If $f, g \in \mathcal{P}[a, b]$, then the Cauchy-Schwarz inequality tells us that

$$|\langle f, g \rangle| \leq \|f\| \|g\|.$$

- We say that a sequence $\{f_N\}_{N \geq 1} \subset \mathcal{P}[a, b]$ converges in mean square sense (MSS) if

$$\lim_{N \rightarrow \infty} \|f_N - f\| = 0.$$

- Know about Fourier series on $[-1, 1]$, relation to Sine and Cosine series through even and odd functions, and item periodic extensions of functions.
- Generalized Fourier series: For a sequence of functions $\{X_k\}_{k=1}^{\infty} \subset \mathcal{P}[a, b] \setminus \{0\}$ that is orthogonal with respect to inner product we define generalized truncated Fourier series

$$S_N(f)(x) := \sum_{k=1}^N c_k X_k(x), \quad \text{with} \quad c_k := \frac{\langle f, X_k \rangle}{\langle X_k, X_k \rangle}$$

- Example: Sine series on $\mathcal{P}[0, 1]$:

$$S_N(f)(x) := \sum_{k=1}^N c_k \sin(k\pi x), \quad \text{with} \quad c_k := \frac{\langle f, \sin(k\pi x) \rangle}{\langle \sin(k\pi x), \sin(k\pi x) \rangle} = 2\langle f, \sin(k\pi x) \rangle.$$

- Result: If $f \in \mathcal{P}[a, b]$, then

$$\langle f - S_N(f), X_k \rangle = 0 \quad \forall k = 1, 2, \dots, N$$

Implication: $f - S_N(f)$ is orthogonal to any function in $\text{Span}(X_1, \dots, X_N)$.

- In particular, $S_N(f) \in \text{Span}(X_1, \dots, X_N)$, and

$$\langle f - S_N(f), S_N(f) \rangle = 0 \implies \langle f, S_N(f) \rangle = \|S_N(f)\|^2 = \sum_{k=1}^N c_k^2 \|X_k\|^2.$$

- Orthogonal decomposition of f : for any $N \geq 0$, f is sum of orthogonal functions

$$f = S_N(f) + (f - S_N(f)) \implies \|f\|^2 = \|S_N(f)\|^2 + \|f - S_N(f)\|^2$$

- Last two points lead to Bessel's inequality: for any $f \in \mathcal{P}[a, b]$

$$\sum_{k=1}^N c_k^2 \|X_k\|^2 = \|S_N(f)\|^2 \leq \|f\|^2$$

- Result: Let $f \in \mathcal{P}[a, b]$. Then $S_N(f) \rightarrow f$ in MSS if and only if

$$\sum_{k=1}^{\infty} c_k^2 \|X_k\|^2 = \|f\|^2 \quad \text{Parseval's identity.}$$

- Special case: For Fourier series on $[-1, 1]$, Parseval's identity holds for any $f \in \mathcal{P}[-1, 1]$.

- Poincare's inequality: Let $f' \in \mathcal{P}[a, b]$ and assume that $f(a) = 0$. Then

$$\int_a^b f^2(x) dx \leq \frac{(b-a)^2}{2} \|f'\|^2$$

Follows from $f(x) = \int_a^x f'(y) dy$ and using Cauchy-Schwarz inequality.

- For sequences of functions $\{f_N\}_{N=1}^{\infty}$, know how to define convergence in mean square sense, pointwise convergence and uniform convergence on an interval $[a, b]$.
- For the Fourier series $S_N(f)$ on $[-1, 1]$ it holds as $N \rightarrow \infty$ that
 - 1 $S_N(f) \rightarrow f$ in MSS if $f \in \mathcal{P}[-1, 1]$,
 - 2 $S_N(f) \rightarrow (f_{\text{per}}(x+) + f_{\text{per}}(x-))/2$ pointwise if $f \in \mathcal{P}[-1, 1]$ and its periodic extension is one-sided differentiable, meaning limits

$$\lim_{h \downarrow 0} \frac{f_{\text{per}}(x-) - f_{\text{per}}(x-h)}{h} \quad \text{and} \quad \lim_{h \downarrow 0} \frac{f_{\text{per}}(x+h) - f_{\text{per}}(x+)}{h}$$

both exist for all $x \in [-1, 1]$. (In short, we expressed this as $f \in \mathcal{P}'[-1, 1]$ in the lecture notes).

- 3 $S_N(f) \rightarrow f$ uniformly if $f' \in \mathcal{P}[-1, 1]$ and $f(-1) = f(1)$.

- Uniform convergence $f_N \rightarrow f$ on $[a, b]$ implies pointwise- and MSS convergence $f_N \rightarrow f$ on $[a, b]$
- Note from previous bullet that when uniform convergence holds, then the limit f is the same for all three cases of convergence.
- For $f \in \mathcal{P}[-1, 1]$, know how to compute Fourier coefficients and use Parseval's identity to translate

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty} a_k^2 + b_k^2 = \|f\|^2$$

into to expressions like

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

(here in the case $f(x) = x$).

- Differentiation of Fourier series on $[-1, 1]$ (from Chapter 8). If $f' \in \mathcal{P}[-1, 1]$ and $f(-1) = f(1)$ with

$$S_{\infty}(f) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\pi x) + b_k \sin(k\pi x),$$

then

$$S_{\infty}(f') = \sum_{k=1}^{\infty} \alpha_k \cos(k\pi x) + \beta_k \sin(k\pi x)$$

with $\alpha_k = k\pi b_k$ and $\beta_k = -k\pi a_k$ (follows by integration by parts).

- By above and Parseval's identity:

$$\|S_{\infty}(f')\|^2 = \sum_{k=1}^{\infty} \pi^2 k^2 (a_k^2 + b_k^2) = \|f'\|^2$$

- Extension to higher order derivatives: Let $m \geq 1$ be an integer. If $f \in C_p^{m-1}[-1, 1]$ and $f^{(m)} \in \mathcal{P}[-1, 1]$, then

$$\sum_{k=1}^{\infty} k^{2m} (a_k^2 + b_k^2) = \pi^{-2m} \|f^{(m)}\|^2$$

- And almost opposite result: Let

$$S_N(x) = \frac{a_0}{2} + \sum_{k=1}^N a_k \cos(k\pi x) + b_k \sin(k\pi x), \quad N = 1, 2, \dots$$

be a sequence of functions with given coefficients $\{a_k\}, \{b_k\} \subset \mathbb{R}$ that for some integer $m \geq 1$ satisfy

$$\sum_{k=1}^{\infty} k^{2m} (a_k^2 + b_k^2) < \infty.$$

Then there exists an $f \in C_p^{m-1}[-1, 1]$ such that

$$S_N^{(j)} \rightarrow f^{(j)} \quad \text{uniformly as } N \rightarrow \infty$$

for all $j \in \{0, \dots, m-1\}$. Furthermore, $S_\infty = S_\infty(f)$, which means that S_∞ is the Fourier series of f .

- Let $f \in C_{p,0}^m[0,1]$ and let $\tilde{f} \in C_p^m[-1,1]$ denote odd extension.
- Restricted to such functions, we showed equivalence between uniform convergence of Fourier series $S_N(\tilde{f})$ on $[-1,1]$ and uniform convergence of Sine series $S_N(f)$ on $[0,1]$.
- Studied the heat equation

$$\begin{aligned}u_t &= u_{xx} & x \in (0,1), t > 0 \\u(0,t) &= u(1,t) = 0, & t > 0 \\u(x,0) &= f(x) & x \in [0,1]\end{aligned}$$

and showed that if $f \in \mathcal{P}[0,1]$ then BC and equation $u_t = u_{xx}$ is satisfied by formal solution.

- For initial condition formal solution $u(x, t)$ satisfies the following:

1 If $f \in \mathcal{P}[0, 1]$, then $\lim_{t \downarrow 0} \|u(\cdot, t) - f\| = 0$.

2 If $f \in C_{p,0}^0[0, 1]$ and $f' \in \mathcal{P}[0, 1]$, then $\lim_{t \downarrow 0} \|u(\cdot, t) - f\|_\infty = 0$.



Implication of point [2]:

$u \in C^{2,1}\left((0, 1) \times (0, T]\right) \cap C([0, 1] \times [0, T])$ and the formal solution is indeed the unique exact solution of the PDE