Plan: • Generalized Fourier series (Sturm-Liouville problems)
• Fourier series on $[0,1]$

For $\left.\begin{aligned} u_t &= u_{xx} && x\in(0,1),\ t>0 \\ u(t,0)&=u(t,1)=0 && t\geq 0 \end{aligned}\right\}$

Ansatz $u(x,t)=X(x)T(t)$ leads to eigenval problem (that we can solve)

$$\begin{aligned} -X''(x) &= \lambda X(x) && x\in(0,1) \\ X(0)&=X(1)=0 \end{aligned}$$

with orthogonal eigfcns (X_k).

Other settings:

• Dirichlet non-zero $u(t,0)=a,\ u(t,1)=b$:

Also $X_k(x)=\sin(k\pi x)$ (see Exercise 3.11)

• Neumann-0 BC: $X_k(x)=\cos(k\pi x)$

• Periodic BC: $X_k(x)=\cos(k\pi x)$, $\tilde{X}_k(x)=\sin(k\pi x)$

New extension: Mixed BC

$$\left.\begin{aligned} u_t &= u_{xx} \\ u(0,t)&=0,\ u_x(1,t)=0 && t>0, \end{aligned}\right\} (2)$$

leads to eigenvalue problem

$$\left.\begin{aligned} \underbrace{-X''(x)}_{=:LX} &= \lambda X(x) && x\in(0,1) \\ X(0)&=0,\ X'(1)=0 \end{aligned}\right\} (3)$$

Seek orthogonal $X_k(x)$ $k = 0, 1, \ldots$
solving (3) (eigenfunctions of L).

That is: $\langle X_k, X_m \rangle = \int_0^1 X_k(x)\, X_m(x)\, dx = 0$ if $k \neq m$

First observe $\int_0^1 -\underbrace{X''(x)}_{\text{II}}\, X(x)\, dx = \lambda \int_0^1 (X(x))^2\, dx$

$\int_0^1 (X'(x))^2\, dx$

$X \not\equiv 0$ & $X(0) = 0 \Rightarrow X' \not\equiv 0 \Rightarrow \lambda = \dfrac{\int_0^1 (X')^2\, dx}{\int_0^1 (X)^2\, dx} > 0.$

General sol: $X(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$

$X(0) = 0 \Rightarrow c_1 = 0$

$0 = X'(1) = c_2 \sqrt{\lambda} \cos(\sqrt{\lambda})$

Set $c_2 = 1$ (any value $\neq 0$ is ok). Then

$\cos(\sqrt{\lambda}) = 0 \Rightarrow \sqrt{\lambda} = (k + \frac{1}{2})\pi$, $k = 0, 1, \ldots$

And $\lambda_k = (k + \frac{1}{2})^2 \pi^2$, $X_k(x) = \sin((k + \frac{1}{2})\pi x)$
are orthogonal eigenfunctions of L:

$$\begin{aligned} \lambda_k \langle X_k, X_m \rangle &= \langle L X_k, X_m \rangle \\ &= \langle X_k, L X_m \rangle \quad \text{(integration by parts)} \\ &= \lambda_m \langle X_k, X_m \rangle, \end{aligned}$$

$\lambda_k \neq \lambda_m \Rightarrow \langle X_k, X_m \rangle = 0.$

$$\left(\langle LX_k, X_m \rangle = \int_0^1 -X_k'' X_m \, dx \right.$$

$$= \left[-X_k' X_m \right]_0^1 + \int_0^1 X_k' X_m' \, dx$$

$$= \int_0^1 X_k (-X_m'') \, dx + \left[X_k X_m' \right]_0^1$$

$$\left. = \langle X_k, LX_m \rangle \right)$$

Particular solutions of (2)

$$u_k(x,t) = T_k(t) X_k(x)$$

$$= e^{-(k+\frac{1}{2})^2 \pi^2 t} \sin\left((k+\tfrac{1}{2})\pi x\right) \qquad k = 0, 1, \ldots$$

(since $T_k' = -\lambda_k T_k$ has sol $T_k(t) = e^{-\lambda_k t}$)

Formal sol of (2): $u(x,t) = \sum_{k=0}^{\infty} c_k u_k(x,t)$

where c_k is determined from boundary data. If $u(x,0) = f(x)$, for example, then

$$c_k = \frac{\langle f, X_k \rangle}{\langle X_k, X_k \rangle}.$$

(See Ex 8.5 in TW for solving $-X'' = \lambda X$ with robin BC $X'(0) = X(0)$, $X'(1) = -X(1)$.) Next week

Sturm-Liouville problems

Extension of heat eq:

$$\left.\begin{aligned} u_t &= (p(x)u_x)_x - q(x)u && x \in (0,1),\ t>0 \\ u(0,t) &= u(1,t) = 0 && t \geq 0 \\ u(x,0) &= f(x) && x \in (0,1) \end{aligned}\right\} (4)$$

With given functions

$$\left.\begin{aligned} p &= p(x) \geq \alpha > 0 \\ q &= q(x) \geq 0 \end{aligned}\right\} \forall x \in [0,1] \text{ with } p', q \in C([0,1]) \quad (5)$$

Particular solutions of (4) by ansatz $u(x,t) = T(t)X(x)$

$$\Rightarrow \frac{-T'(t)}{T(t)} = \frac{-(p(x)X')' + q(x)X(x)}{X(x)} = \lambda$$

Eigenval problem: seek $X \in C_0^2((0,1)) \setminus \{0\}$ s.t.

$$\underbrace{-(pX')' + qX}_{=:LX} = \lambda X \qquad \forall x \in (0,1) \quad (6)$$

(6) is called a Sturm-Liouville problem (with Dirichlet 0-valued BC).

It holds $\forall X, Y \in C_0^2(0,1)$ that

(a) $\langle LX, Y\rangle = \langle X, LY\rangle$ (symmetric)

(b) $\langle LX, X\rangle \geq 0$
with $\langle LX, X\rangle = 0 \Leftrightarrow X \equiv 0$ (pos def.)

Proof of (b):

$$\langle LX, X\rangle = \int_0^1 -(pX')'X + qX^2\,dx$$

$$= \underbrace{\int_0^1 q(x)X^2\,dx}_{\geq 0} - \underbrace{[pX'X]_0^1}_{=0} + \int_0^1 p(x)(X')^2\,dx$$

$$\overset{(5)}{\geq} \alpha \int_0^1 (X'(x))^2\,dx = \alpha\langle X', X'\rangle \geq 0$$

And $(\langle X', X'\rangle = 0 \ \& X(0)=0) \Leftrightarrow X \equiv 0$.

Consequences:

- If (λ_k, X_k) and (λ_m, X_m) are eigenpairs of (6) with $\lambda_k \neq \lambda_m$ then $\langle X_k, X_m\rangle = 0$ as

$$\lambda_k \langle X_k, X_m \rangle = \langle LX_k, X_m \rangle = \langle X_m, LX_m \rangle = \lambda_m \langle X_k, X_m \rangle$$

$$\Rightarrow (\lambda_k - \lambda_m) \langle X_k, X_m \rangle = 0$$

- $\lambda > 0$ in (6), since if (λ, X) is sol, then

$$\lambda \langle X, X \rangle = \langle LX, X \rangle > 0 \quad (\text{since } X \not\equiv 0)$$

$$\Rightarrow \lambda = \frac{\langle LX, X \rangle}{\langle X, X \rangle} > 0.$$

Theorem 1 When (5) holds, there exists solutions (λ_k, X_k) to (6) with

$$0 < \lambda_1 < \lambda_2 < \cdots, \quad \lim_{k \to \infty} \lambda_k = \infty$$

and X_k orthogonal.

Ex When $p(x) = (1+x)$ and $q(x) = 0$, then

$$-((1+x)X')' = \lambda X$$
$$X(0) = X(1) = 0$$

has solutions

$$X_k(x) = (1+x)^{-\frac{1}{2}} \sin\left[k\pi \frac{\log(1+x)}{\log(2)}\right]$$

$$\lambda_k = \left(\frac{k\pi}{\log(2)}\right)^2 + \frac{1}{4}.$$ (See Exercise 8.15)

Fourier series on $[-1,1]$:

Recall $\{\cos(k\pi x)\}_{k=0}^{\infty} \cup \{\sin(k\pi x)\}_{k=1}^{\infty}$ is orthogonal with respect to inner product

$$\langle u, v\rangle = \int_{-1}^{1} u(x)v(x)\,dx \qquad \text{(Exercise 3.15)}$$

and Fourier series of $f:[-1,1]\to\mathbb{R}$ is

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(k\pi x) + b_k \sin(k\pi x)\right)$$

where $a_k = \langle f, \cos(k\pi x)\rangle$

$b_k = \langle f, \sin(k\pi x)\rangle$

"$\sim$" denotes formally equal.

Truncated Fourier series

$$S_N(f)(x) := \frac{a_0}{2} + \sum_{k=1}^{N} \left(a_k \cos(k\pi x) + b_k \sin(k\pi x)\right)$$

Note: $f \sim S_\infty(f)$, but this does not mean that $f(x) = S_\infty(f)(x)$ $\forall x \in [0,1]$:

Ex For $f(x) = x$,

$a_k = \int_{-1}^{1} \cos(k\pi x)\, x\, dx = 0$ &

$b_k = \int_{-1}^{1} \sin(k\pi x)\, x\, dx = \frac{2}{k\pi}(-1)^{k+1}$ yielding

$$f(x) \sim \sum_{k=1}^{\infty} \frac{2}{k\pi}(-1)^{k+1} \sin(k\pi x) = S_\infty(f)(x)$$

Note that $f(-1) = 1$, while

$$S_\infty(f)(1) = \sum_{k=1}^{\infty} \frac{2}{k\pi} (-1)^{k+1} \cdot \underbrace{\sin(-k\pi)}_{=0} = 0$$

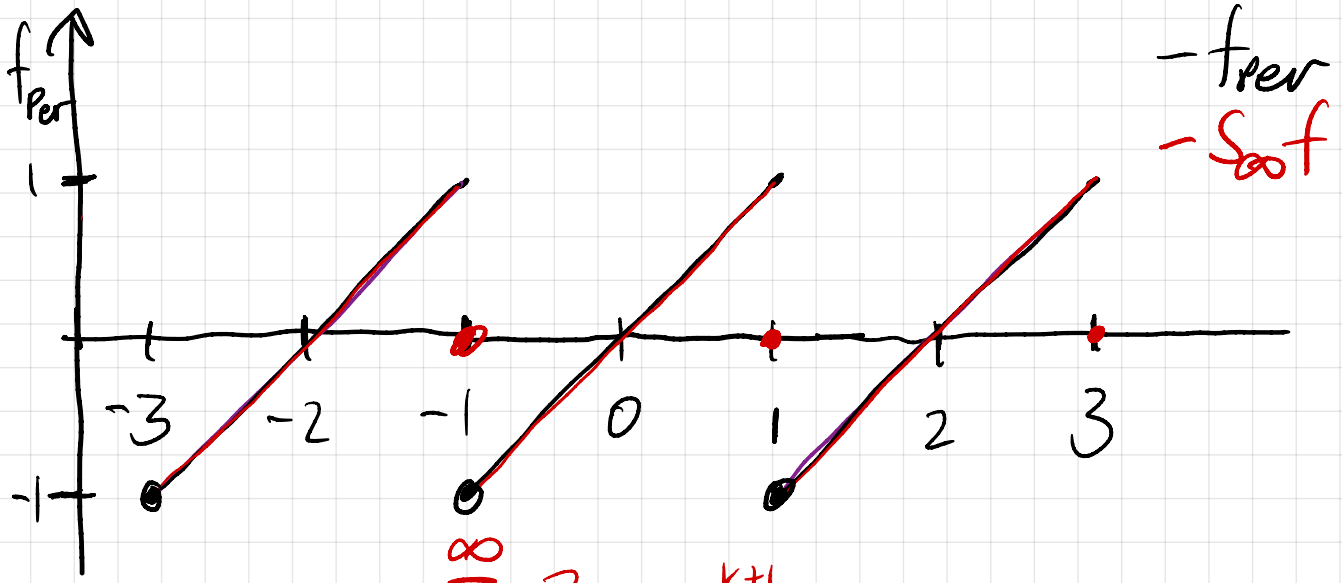
Why $f(1) \neq S_\infty(1)$ follows from periodic extensions:

Any function $f: [-1,1] \to \mathbb{R}$ can be 2-periodically extended from $[-1,1)$ to $\mathbb{R}$ by

$$f_{per}(x+2k) := f(x) \quad \forall x \in [-1,1) \ \& \ k \in \mathbb{Z}$$

$$\Rightarrow f_{per}(x) = f_{per}(x+2) \quad \forall x \in \mathbb{R}$$

For $f(x) = x$, we get



$$S_\infty(f)(x) = \sum_{k=1}^{\infty} \frac{2}{k\pi} (-1)^{k+1} \sin(k\pi x)$$

is 2-periodic by construction with

$$S_\infty(f)(-1) = \frac{f(-1)+f(1)}{2} \neq f(-1) \text{ when } f(-1) \neq f(1).$$

↑ Problem

Goals today:

- Properties of $S_\infty(f)$: even/odd parts, differentiation
- Next lectures: mixed boundary val problems, convergence $S_N(f) \to f$ (when and how does it hold?)

Def We say f on $[a,b]$ is

- periodic if $f(-a) = f(b)$ and
- piecewise continuous if f is continuous at all but a finite number of points $\{x_i\}_{i=0}^{n} \subset [-a,b]$. And the limits

$$\lim_{x \uparrow x_i} f(x) =: f(x_i-) \quad \& \quad \lim_{x \downarrow x_i} f(x) =: f(x_i+)$$

exists for $i = 0, \ldots, n$.

f is called even if $f(x) = f(-x)$

and odd if $f(x) = -f(-x)$ $\quad \forall x \in [-1, 1]$

Ex

even: $1, x^2, x^4, \ldots, \cos(k\pi x)$

odd: $x, x^3, \sin(k\pi x)$

Property If f is odd, then

$$f(x) \sim \sum_{k=1}^{\infty} b_k \sin(k\pi x)$$

and if f is even, then

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\pi x)$$

Proof odd f:

$$a_k = \int_{-1}^{1} f(x) \cos(k\pi x)\, dx$$

$$= \int_{-1}^{0} f(x) \cos(k\pi x)\, dx + \int_{0}^{1} f(x) \cos(k\pi x)\, dx \qquad (Y = -x)$$

$$= \int_{0}^{1} \underbrace{f(-Y)}_{= -f(Y)} \cos(-k\pi Y)\, dY + \int_{0}^{1} f(x) \cos(k\pi x)\, dx$$

$$= 0$$

∎

For a function $\tilde{f}$ on $[0,1]$ with sine series

$$\sum_{k=1}^{\infty} \tilde{b}_k \sin(k\pi x), \quad \tilde{b}_k = 2\int_0^1 \tilde{f}(x)\sin(k\pi x)\,dx,$$

Then the odd extension on $[-1,1]$:

$$f(x) := \begin{cases} \tilde{f}(x) & x \in [0,1] \\ -\tilde{f}(-x) & x \in [-1,0) \end{cases}$$

satisfies

$$S_\infty(f) = \sum_{k=1}^{\infty} b_k \sin(k\pi x) \quad \text{with } b_k = \tilde{b}_k \text{ as}$$

$$b_k = \int_{-1}^{1} f(x)\sin(k\pi x)\,dx = 2\int_0^1 \tilde{f}(x)\sin(k\pi x)\,dx$$

So when f is odd, then

$$\boxed{S_\infty(f) = \text{Sine series of } \tilde{f} = f|_{[0,1]}}$$

Application: Let $f = \text{sign}(x) = \begin{cases} 1 & x \geq 1 \\ -1 & x < 0 \end{cases}$

Then f is odd extension of

$$f|_{[0,1]} \equiv 1$$

1 has sine series $\sum_{k=1}^{\infty} \overbrace{\frac{4}{\pi(2k-1)}}^{\tilde{b}_{2k-1}} \sin((2k-1)\pi x)$ for $x \in [0,1]$

$$\Rightarrow \text{sign}(x) \sim \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)} \sin((2k-1)\pi x) \qquad x \in [-1,1]$$

Similarly, if f is even, then

$S_\infty(f) =$ cosine series of $f|_{[0,1]}$.

Theorem (differentiation of Fourier series)

Assume that $f \in C([-1,1])$, f' is piecewise cont. on $[-1,1]$, and $f(-1) = f(1)$. Let

$$S_\infty(f)(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(k\pi x) + b_k \sin(k\pi x)\right)$$

and

$$S_\infty(f')(x) = \frac{\alpha_0}{2} + \sum_{k=1}^{\infty} \left(\alpha_k \cos(k\pi x) + \beta_k \sin(k\pi x)\right)$$

Then $\alpha_0 = 0$ and $\left.\begin{aligned} \alpha_k &= k\pi b_k \\ \beta_k &= -k\pi a_k \end{aligned}\right\} k \geq 1.$

Means that $S_\infty(f)'(x) = S_\infty(f')(x)$

Proof $\alpha_0 = \int_{-1}^{1} f'(x)\,dx = f(1) - f(-1) = 0,$

$$\alpha_k = \int_{-1}^{1} f'(x)\cos(k\pi x)\,dx$$

$$= \left[f(x)\cos(k\pi x)\right]_{-1}^{1} + k\pi \int_{-1}^{1} f(x)\sin(k\pi x)\,dx$$

$$= \underbrace{(-1)^k\left(f(1) - f(-1)\right)}_{=0} + k\pi\, b_k$$

B_k is computed similarly. ∎

Application

For $f(x) = |x|$, let us find $S_\infty(f)$.

Since $f'(x) = \text{sign}(x)$, we know that

$$S_\infty(f')(x) = \sum_{k=1}^{\infty} \underbrace{\frac{4}{\pi(2k-1)}}_{B_{2k-1}} \sin((2k-1)\pi x)$$

Then $a_{2k-1} = \frac{-B_{2k-1}}{\pi(2k-1)} = \frac{-4}{\pi^2(2k-1)^2}$

and $a_0 = \int_{-1}^{1} |x|\,dx = 1$

$$\Rightarrow S_\infty(f)(x) = \frac{1}{2} - \sum_{k=1}^{\infty} \frac{4}{\pi^2(2k-1)^2} \cos((2k-1)\pi x)$$

Let $\mathcal{P} = \{ f : [a,b] \to \mathbb{R} \mid f \text{ is piecewise continuous on } [a,b] \}$

- $\langle f, g \rangle := \int_a^b f(x)\, g(x)\, dx \quad \forall f, g \in \mathcal{P}$
- Semi-norm:

$$\|f\| := \sqrt{\langle f, f \rangle} = \sqrt{\int_a^b (f(x))^2\, dx}$$

and mean square distance

$$\|f - g\| = \sqrt{\int_a^b (f(x) - g(x))^2\, dx}$$

It holds that

a) $\|\alpha f\| = |\alpha| \, \|f\| \quad \forall \alpha \in \mathbb{R} \ \& \ f \in \mathcal{P}$

b) $\|f + g\| \leq \|f\| + \|g\| \quad \forall f, g \in \mathcal{P}$ (will show)

c) $\|f\| \geq 0$ and

$\|f\| = 0 \Leftrightarrow f(x) = 0$ for almost all $x \in [a,b]$

(is norm on $\mathcal{P}$ if we relax "=" in $u = v$ for $u, v \in \mathcal{P}$ to mean $u(x) = v(x)$ for almost all $x \in [a,b]$.)

Def 1 A sequence $\{f_N\}_{N=1}^{\infty} \subset \mathcal{P}$ is said to converge in mean square sense (MSS) to $f \in \mathcal{P}$ if

$$\lim_{N\to\infty} \|f_N - f\| = 0.$$

Example 1 Let $[a,b] = [0,1]$ and $f_N(x) = x^N$. Then $f_N \xrightarrow{N\to\infty} f \equiv 0$ in MSS, since

$$\|f_N - 0\|^2 = \int_0^1 (f_N(x))^2 dx = \int_0^1 x^{2N} dx$$

$$= \frac{1}{2N+1} \to 0 \quad \text{as } N \to \infty.$$

NB! Convergence in MSS is not equivalent to pointwise convergence

In Example 1, $f_N(x) = x^N \to \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases}$

hence $f_N(1) \not\to 0$.

Lemma 1 (Cauchy - Schwarz ineq)
If $f, g \in \mathcal{P}$, then

$$|\langle f, g \rangle| \le \|f\| \|g\| \qquad (1)$$

Proof If $f(x) = 0$ for all $x \in [a,b]$, then (1) becomes $0 \le 0$, which holds.

Assume next $\|f\| > 0$ and let

$$P(t) := \int_a^b (tf(x) - g(x))^2 \, dx \qquad \text{for } t \in \mathbb{R}$$

$$= t^2 \int_a^b f^2(x)\,dx - 2t \int_a^b f(x)g(x)\,dx + \int_a^b g^2(x)\,dx$$

$$= t^2 \|f\|^2 - 2t \langle f, g \rangle + \|g\|^2$$

Note: $P(t)$ is strictly convex $(P'' = \|f\|^2 > 0)$, with global min at t^* where $P'(t^*) = 0$.

Namely,

$$t^* = \frac{\langle f, g \rangle}{\|f\|^2}, \text{ and}$$

$$0 \le P(t^*) = \frac{\langle f, g \rangle^2 - 2\langle f, g \rangle^2}{\|f\|^2} + \|g\|^2$$

$$\Rightarrow |\langle f, g \rangle| \le \|f\| \, \|g\|$$

∎

Use of Cauchy-Schwarz

Verification of triangle ineq: For $f, g \in \mathcal{P}$

$$\|f+g\|^2 = \langle f+g, f+g\rangle$$

$$= \langle f, f\rangle + 2\langle f, g\rangle + \langle g, g\rangle$$

$$\overset{\text{Cauchy-Schwarz}}{\leq} \|f\|^2 + 2\|f\|\|g\| + \|g\|^2$$

$$= (\|f\| + \|g\|)^2$$

taking $\sqrt{\ }$ both sides $\Rightarrow$ $\|f+g\| \leq \|f\| + \|g\|$.

Poincaré inequality

Lemma 2 Let $f \in C[a,b]$ satisfy that f' is piecewise continuous and $f(a) = 0$.

Then $\|f\| \leq \frac{(b-a)}{\sqrt{2}} \|f'\|$.

Proof For any $x \in [a,b]$, $f(x) = \underbrace{f(a)}_{=0} + \int_a^x f'(y)\,dy$

$$\Rightarrow |f(x)|^2 = \left(\int_a^x 1 \cdot f'(y)\,dy\right)^2 \quad \left(\text{Cauchy-S on } [a,x]\right)$$

$$\leq \left(\int_a^x 1^2\,dy\right)\left(\int_a^x (f'(y))^2\,dy\right)$$
$$\leq (x-a)\int_a^b (f'(y))^2\,dy = (x-a)\,\|f'\|^2$$

Hence $\|f\|^2 = \int_a^b |f(x)|^2\,dx$

$$\leq \int_a^b (x-a)\,dx \cdot \|f'\|^2$$
$$\leq \frac{(b-a)^2}{2}\|f'\|^2$$

∎

Sharpening of energy estimate

Recall that for $u_t = u_{xx}$ $\quad t>0,\ x\in(0,1)$

$u(0,t) = u(1,t) = 0 \quad t\geq 0,$

we showed for $E(t) := \int_0^1 u^2(x,t)\,dx$

that $E'(t) = -2\int_0^1 u_x^2(x,t)\,dx \leq 0, \quad (3)$

and thereby $E(t) \leq E(0) \quad t\geq 0$.

Sharper upper bound $E(t)$:

Since $u(0,t)=0$ and $u(\cdot,t)\in C^2$,

Lemma 2 on $[a,b]=[0,1]$ yields

$$E(t) = \int_0^1 u^2(x,t)\,dx = \|u(\cdot,t)\|^2$$

$$\overset{\text{Lem 3}}{\leq} \frac{1}{2}\|u_x(\cdot,t)\|^2 = \frac{1}{2}\underbrace{\int_0^1 u_x^2(x,t)\,dx}_{=-\frac{1}{2}E'(t)}$$

$$\overset{(3)}{=} -\frac{1}{4}E'(t).$$

Hence $E'(t) \leq -4E(t) \quad | \cdot e^{4t}$

$$\Rightarrow \quad \left(e^{4t}E(t)\right)' \leq 0$$

$$\Rightarrow \quad E(t) \leq e^{-4t}E(0) \qquad t \geq 0$$

and $\lim_{t\to\infty} E(t) = 0.$

Exercise 7.9

$$\left.\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ \frac{\partial u}{\partial n} &= g && \text{on } \partial\Omega \end{aligned}\right\} (*)$$

a) Show that

$\iint_\Omega f \, dx\,dy = -\int_{\partial\Omega} g \, ds$ is necessary condition for $(*)$ to have a solution.

Sol: If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ solves $(*)$, then

$$\iint_\Omega f \, dx\,dy = \iint_\Omega -\Delta u \, dx\,dy$$

$$= -\iint_\Omega \operatorname{div} \nabla u \, dx\,dy$$

$$\overset{\text{div thm}}{=} -\int_{\partial\Omega} \frac{\partial u}{\partial n} \, ds$$

$$= -\int_{\partial\Omega} g \, ds.$$

b) Show that (*) has at most one solution satisfying

$$\iint_\Omega u \, dx\,dy = 0. \qquad (**)$$

Why is this extra cond. needed?

Sol: Assume $u, v \in C^2(\Omega) \cap C^1(\bar{\Omega})$ solve (*) and satisfy (**)

Then $w = u - v$ solves

$$-\Delta w = 0 \quad \text{in } \Omega$$

$$\frac{\partial w}{\partial n} = 0 \quad \text{on } \partial\Omega$$

And

$$0 = \iint_\Omega (-\Delta w)\, w \, dx\,dy$$

$$= \iint_\Omega (w_x)^2 + (w_y)^2 \, dx\,dy - \underbrace{\int_{\partial\Omega} w \underbrace{\frac{\partial w}{\partial n}}_{=0} ds}_{\equiv 0}$$

$$\Rightarrow w_x \equiv 0 \;\&\; w_y \equiv 0$$

$$\Rightarrow w \equiv c. \text{ And since}$$

$$c \, \text{Area}(\Omega) = \iint_\Omega w \, dx\,dy = \iint_\Omega u \, dx\,dy - \iint_\Omega v \, dx\,dy = 0$$

$\Rightarrow C = 0$. Hence $W \equiv 0 \Rightarrow u = v$.

Ex 7.10

$$-\Delta u = f \quad \text{in } \Omega$$

$$\frac{\partial u}{\partial n} + \alpha u = g \quad \text{on } \partial\Omega \quad (\text{Robin BC})$$

with $\alpha > 0$. Show that PDE has at most one sol.

Sol: Assume $u, v \in C^2(\Omega) \cap C^1(\bar{\Omega})$ solve PDE. Then

$w = u - v$ solve

$$-\Delta w = 0 \quad \text{in } \Omega$$

$$\frac{\partial w}{\partial n} + \alpha w = 0 \quad \text{on } \partial\Omega$$

And

$$0 = \iint_\Omega (-\Delta w) w \, dx dy$$

$$= \iint_\Omega |\nabla w|^2 dx dy - \int_{\partial\Omega} w \frac{\partial w}{\partial n} \, ds$$

$$= \iint |\nabla w|^2 dx dy + \alpha \int_{\partial\Omega} w^2 \, ds$$

$\Rightarrow w_x \equiv 0, w_y \equiv 0$ in Ω

& $w = 0$ on $\partial\Omega$

$\Rightarrow w \equiv 0 \Rightarrow u = v$.

Why needed: Sol of (*) only determined up to const.

Ex 7.17 Let $\Omega = (0,1)^2$ &

Consider eigval problem: find (λ, u), with $\lambda \in \mathbb{R}, u \in C_0^2(\Omega) \setminus \{0\}$ s.t.

$$-\Delta u = \lambda u \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

a) Show that $\lambda > 0$. Sol:

1. $\iint_\Omega (-\Delta u)\, u \, dx dy = \iint_\Omega |\nabla u|^2 dx dy - \int_{\partial\Omega} u \frac{\partial u}{\partial n} ds$

& 2. $\iint_\Omega (-\Delta u) u \, dx dy = \lambda \iint_\Omega u^2 dx dy$

$$\Rightarrow \lambda = \frac{\iint_\Omega |\nabla u|^2 dx dy}{\iint_\Omega u^2 dx dy} > 0$$

Since $(u \not\equiv 0$ and $u|_{\partial\Omega} = 0) \Rightarrow |\nabla u| \neq 0$.

b) If for each $x \in (0,1)$,

$$u(x,y) = \sum_{k=1}^{\infty} a_k(x) \sin(k\pi y),$$

explain how this leads to eigenfuns

$$u_{jk}(x,y) = \sin(j\pi x)\sin(k\pi y)$$

with $\lambda_{jk} = (j^2+k^2)\pi^2$ $\quad j,k \geq 1$

Sol: Let us write a_k as a sine series

$$a_k(x) = \sum_{j=1}^{\infty} c_{jk} \sin(j\pi x)$$

$$c_{jk} = 2\langle a_k(x), \sin(k\pi x)\rangle$$

Then formally

$$u(x,y) = \sum_{j,k=1}^{\infty} c_{jk} \underbrace{\sin(j\pi x)\sin(k\pi y)}_{u_{jk}}$$

& $-\Delta u_{jk} = (j^2+k^2)\pi^2 u_{jk}$.

7.19 Consider 2D heat eq.

$u_t = \Delta u$ in $\Omega, t \geq 0$

$u(x,y,t) = 0$ for $(x,y) \in \partial\Omega$

$u(x,y,0) = f(x,y)$ $(x,y) \in \Omega$

a) Show that for sufficiently smooth sol,

$$E(t) = \iint_\Omega u^2(x,y,t)\,dx\,dy \leq \iint_\Omega f^2(x,y)\,dx\,dy$$

for $t \geq 0$.

Sol: (for $u \in C^{2,1}(\bar{\Omega} \times [0,\infty))$

$$E'(t) = \iint_\Omega \frac{d}{dt} u^2\,dx\,dy$$

$$= 2\iint_\Omega u_t u\,dx\,dy$$

$$= 2\iint_\Omega \Delta u\, u\,dx\,dy$$

div thm

$$= -2\iint_\Omega |\nabla u|^2\,dx\,dy - \underbrace{\int_{\partial\Omega} u \frac{\partial u}{\partial n}\,ds}_{=0}$$

($u = 0$ on $\partial\Omega$)

$$\leq 0$$

$$\Rightarrow E(t) \leq E(0) = \iint_\Omega f^2(x,y)\,dx\,dy$$

$\forall\ t \geq 0$.

(b) If $u, v \in C^{2,1}(\bar{\Omega} \times [0,\infty))$ both solve PDE, then $w = u - v$ solves

$$w_t = \Delta w \quad \text{in } \Omega \times (0,\infty)$$
$$w = 0 \quad \text{on } \partial\Omega \times [0,\infty)$$
$$w = 0 \quad \text{on } \Omega \times \{0\}$$

By energy arg

$$\iint_\Omega w^2(x,y,t)\,dx\,dy \le \iint_\Omega w^2(x,y,0)\,dx\,dy = 0$$

$\Rightarrow w \equiv 0 \Rightarrow u = v.$

c) For $\Omega = (0,1)$, use Ex 7.17 b) to construct formal sol.

Ansatz $V_{jk}(x,y,t) = T_{jk}(t)\,u_{jk}(x,y)$

leads to $(\partial_t V_{jk} = \Delta V_{jk})$

$$\frac{T_{jk}'(t)}{T_{jk}(t)} = \frac{\Delta u_{jk}}{u_{jk}} = -\lambda_{jk}$$

with $u_{jk}|_{\partial\Omega} = 0$. From 7.(7 b)

$u_{jk} = \sin(j\pi x)\sin(k\pi y)$, $\lambda_{jk} = (j^2+k^2)\pi^2$

$$T_{jk}(t) = e^{-\lambda_{jk} t} = e^{-(j^2+k^2)\pi^2 t}$$

Formal sol

$$u(x,y,t) = \sum_{j,k=1}^{\infty} c_{jk} e^{-(j^2+k^2)\pi^2 t} \sin(j\pi x)\sin(k\pi y)$$

where $u(x,y,0) = f(x,y)$

$$\Rightarrow$$

$$c_{jk} = \frac{\int_0^1\int_0^1 f(x,y)\sin(j\pi x)\sin(k\pi y)\,dx\,dy}{\int_0^1\int_0^1 \sin^2(j\pi x)\sin^2(k\pi y)\,dx\,dy}$$

$$=: \frac{\langle f, u_{jk}\rangle}{\langle u_{jk}, u_{jk}\rangle}$$

$$= 4\langle f, u_{jk}\rangle .$$

Ex 7.16

$$-\Delta u = 2x(1-x) + 2y(1-y) \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

a) Show that
$u = x(1-x)y(1-y)$ is exact sol. in $C_0^2(\Omega)$

Sol: Know that $L = -\Delta$ is pos def on $C_0^2(\Omega)$, so sol is unique.

$$u_{xx} = -2y(1-y) \quad \& \quad u_{yy} = -2x(1-x)$$
$$\Rightarrow -\Delta u = \underbrace{2x(1-x) + 2y(1-y)}_{f(x,y)}$$
$$\& \; u|_{\partial\Omega} = 0$$

Extra: Show that numerical solution (7.33) agrees with exact solution in all mesh points.

Sol: Let $\tau_h(x,y) = (L_h u)(x,y) - f(x,y)$ for $(x,y) \in \Omega_h$. Know that

$$\|\tau_h\|_{h,\infty} \leq \frac{(\|u_{xxxx}\|_\infty + \|u_{yyyy}\|_\infty)\, h^2}{6} = 0$$

as $u_{xxx} = 0$ & $u_{yyy} = 0$

Let $v \in D_{h,0}$ solve $L_h v = f$.

Then

$$\|u - v\|_{h,\infty} \leq \frac{1}{8} \|\tau_h\|_{h,\infty} = 0.$$

(As $L_h(u-v) = \tau_h$ in Ω_h

$u - v = 0$ on $\partial\Omega_h$

And Prop 6.1 $\Rightarrow$ $\|u - v\|_{h,\infty} \leq \frac{1}{8} \|\tau_h\| = 0$.)