Recall $P[a,b]$ is set of piecewise continuous functions on $[a,b]$.

Shorthand $P = P[a,b]$, when interval is given.

Def Let $\{X_k\}_{k=1}^{\infty} \subset P$ be:

- orthogonal $\langle X_k, X_m \rangle = \int_a^b X_k(x) X_m(x)\,dx = 0 \quad k \neq m$
- and non-zero $\|X_k\| \neq 0 \quad \forall k$.

Then for any $f \in P$, the function

$$S_\infty(f)(x) = \sum_{k=1}^{\infty} \underbrace{\frac{\langle f, X_k \rangle}{\|X_k\|^2}}_{=:c_k} X_k(x), \quad x \in [a,b]$$

is called the general Fourier series of f.

Notation: $S_N(f) := \sum_{k=1}^{N} c_k X_k \in P$

(truncated general Fourier series)

Examples

- Sine series on $[0,1]$ $\quad X_k(x) = \sin(k\pi x) \quad k \geq 1$
- Example 8.5 TW on $[0,1]$ $\quad X_k(x) = \sin\left((k+\frac{1}{2})\pi x\right) \quad k \geq 0$
- Fourier series on $[-1,1]$: $X_0 \equiv 1$

$$\left.\begin{aligned} X_{2k-1}(x) &= \sin(k\pi x) \\ X_{2k}(x) &= \cos(k\pi x) \end{aligned}\right\} \; k \geq 1$$

Properties (of general Fourier series)

For any function $P_N(x) = \sum_{k=1}^{N} \alpha_k \Sigma_k(x)$ with $\alpha_k \in \mathbb{R}$, it holds that $P_N \in \mathcal{P}$ and

$$\|P_N\|^2 = \sum_{k=1}^{N} \alpha_k^2 \|\Sigma_k\|^2 \qquad (1), \text{ since}$$

$$\|P_N\|^2 = \left\langle \sum_{j=1}^{N} \alpha_j \Sigma_j \,,\, \sum_{k=1}^{N} \alpha_k \Sigma_k \right\rangle$$

$$= \sum_{j=1}^{N} \sum_{k=1}^{N} \alpha_j \alpha_k \langle \Sigma_j, \Sigma_k \rangle \qquad \text{(orthogonality)}$$

$$= \sum_{k=1}^{N} \alpha_k^2 \|\Sigma_k\|^2 .$$

Lemma 1 Let $f \in \mathcal{P}$. Then for any function $P_N = \sum_{k=1}^{N} \alpha_k \Sigma_k$ with $\alpha_k \in \mathbb{R}$, it holds that

$$\langle f - S_N(f), P_N \rangle = 0 \qquad \text{(orthogonality property)}$$

Proof If $1 \le k \le N$, then

$$\langle S_N(f), \Sigma_k \rangle = \left\langle \sum_{j=1}^{N} c_j \Sigma_j, \Sigma_k \right\rangle$$

$$= \sum_{j=1}^{N} c_j \langle \Sigma_j, \Sigma_k \rangle$$

$$= c_k \langle \Sigma_k, \Sigma_k \rangle = \frac{\langle f, \Sigma_k \rangle}{\langle \Sigma_k, \Sigma_k \rangle} \langle \Sigma_k, \Sigma_k \rangle$$

Hence, $\langle f - S_N(f), X_k \rangle = \langle f, X_k \rangle - \langle S_N(f), X_k \rangle$

$= 0$ for $k = 1, \dots, N$,

and by linearity

$$\langle f - S_N(f), P_N \rangle = \left\langle f - S_N(f), \sum_{k=1}^{N} \alpha_k X_k \right\rangle$$

$$= \sum_{k=1}^{N} \alpha_k \underbrace{\langle f - S_N(f), X_k \rangle}_{=0} = 0$$

□

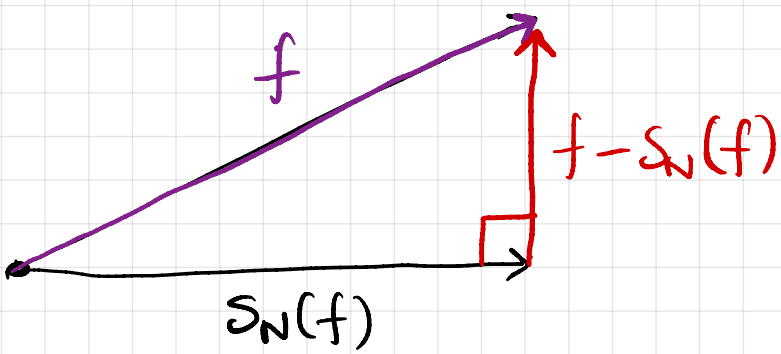
Taking $P_N = S_N(f)$ in Lemma 1 yields $\langle f - S_N(f), S_N(f) \rangle = 0$.

So $(f - S_N(f))$ and $S_N(f)$ are orthogonal, and

$$\langle f, S_N(f) \rangle = \|S_N(f)\|^2 \overset{\text{(i) with } P_N = S_N(f)}{=} \sum_{k=1}^{N} c_k^2 \|X_k\|^2.$$

Orthogonal decomposition

$$f = S_N(f) + (f - S_N(f))$$



From

$$\|f\|^2 = \langle S_N(f) + (f - S_N(f)),\ S_N(f) + (f - S_N(f)) \rangle$$

$$= \|S_N(f)\|^2 + 2\underbrace{\langle f - S_N(f),\ S_N(f) \rangle}_{=0} + \|f - S_N(f)\|^2, \quad (2)$$

we obtain Pythagoras theorem

$$\|f\|^2 \overset{(2)}{=} \|S_N(f)\|^2 + \|f - S_N(f)\|^2 \qquad N \geq 1,$$

and that

$$\|(S_N f)\|^2 \overset{(2)}{\leq} \|f\|^2. \text{ That is}$$

$$\sum_{k=1}^{N} c_k^2 \|X_k\|^2 \leq \|f\|^2 \qquad N \geq 1.$$

Limit $N \to \infty$ yields <u>Bessel's ineq</u>:

$$\sum_{k=1}^{\infty} c_k^2 \|X_k\|^2 \leq \|f\|^2 \qquad (3)$$

Result: $(3) \Rightarrow \lim_{N \to \infty} \|f - S_N(f)\| = 0$

if and only if

$$\sum_{k=1}^{\infty} c_k^2 \|X_k\|^2 = \|f\|^2 \quad (4) \text{ holds}$$

(4) is called Parseval's identity.

and for standard Fourier series, Parseval's id holds for all $f \in \mathcal{P}[-1,1]$

(In general $\{X_k\}$ must be dense in a complete space.)

Examples

Fourier series on $[-1,1]$: Let $f \in \mathcal{P}$ and

$$S_\infty(f) = \underbrace{\frac{a_0}{2}}_{c_0} \cdot \underbrace{1}_{X_0} + \sum_{k=1}^{\infty} \underbrace{a_k}_{c_{2k}} \underbrace{\cos(k\pi x)}_{X_{2k}} + \underbrace{b_k}_{c_{2k-1}} \underbrace{\sin(k\pi x)}_{X_{2k-1}}$$

$$= \sum_{k=0}^{\infty} c_k X_k$$

is nonzero: $\|X_0\|^2 = \int_{-1}^{1} 1^2 dx = 2$, $\|X_k\|^2 = 1$

And Bessel's ineq:

$$\sum_{k=0}^{\infty} c_k^2 \|X_k\|^2 = 2c_0^2 + \sum_{k=1}^{\infty} c_k^2 \leq \|f\|^2$$

Equivalently: $\frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2) \leq \int_{-1}^{1} f^2(x)\, dx$

For sine series on $[0,1]$ and $f \in \mathcal{P}$, we have

$$S_\infty(f) = \sum_{k=1}^{\infty} c_k \underbrace{\sin(k\pi x)}_{\Sigma_k(x)}$$

$$\|\Sigma_k\|^2 = \int_0^1 \sin^2(k\pi x)\,dx = \frac{1}{2}.$$

Bessel's ineq:

$$\sum_{k=1}^{\infty} c_k^2 \|\Sigma_k\|^2 \le \|f\|^2$$

$$\Rightarrow \quad \sum_{k=1}^{\infty} c_k^2 \le 2\int_0^1 f^2(x)\,dx.$$

Application: For $f(x) = x$ on $[0,1]$, one has

$$c_k = 2\langle f, \sin(k\pi x)\rangle = \frac{2}{\pi}\frac{(-1)^{k+1}}{k}$$

and $S_\infty(f) = \sum_{k=1}^{\infty} c_k \sin(k\pi x)$

Bessel's ineq

$$\sum_{k=1}^{\infty} c_k^2 = \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2} \le 2\int_0^1 \underbrace{x^2}_{f^2(x)}\,dx = \frac{2}{3}$$

$$\Rightarrow \sum_{k=1}^{\infty}\frac{1}{k^2} \le \frac{\pi^2}{6}.$$

(Parseval will show it's an equality)

Convergence of functions

For $\{f_N\} \subseteq \mathcal{P}[a,b]$ and $f \in \mathcal{P}$, recall following types of convergence:

- Mean square sense (MSS) on $[a,b]$ if

$$\lim_{N\to\infty} \|f_N - f\| = 0, \quad \text{where } \|f\| = \sqrt{\int_a^b f^2(x)\,dx}$$

- Uniformly on $[a,b]$ if $\lim_{N\to\infty} \|f_N - f\|_\infty = 0$,

where $\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|$

- Pointwise on open/closed set $I \subseteq [a,b]$ if

$$\lim_{N\to\infty} f_N(x) = f(x) \quad \text{for any } x \in I$$

(or equivalently $\lim_{N\to\infty} |f_N(x) - f(x)| = 0$)

Ex $f_N(x) = e^{-Nx}$ converges in MSS to $f \equiv 0$ on $[0,1]$:

$$\|f_N - f\|^2 = \int_0^1 (e^{-Nx} - 0)^2\,dx = \frac{1-e^{-2N}}{2N} \to 0 \text{ as } N\to\infty.$$

And since

$$\lim_{N\to\infty} f_N(x) = \lim_{N\to\infty} e^{-Nx} = \begin{cases} 1 & \text{if } x=0 \\ 0 & \text{if } x\in(0,1] \end{cases},$$

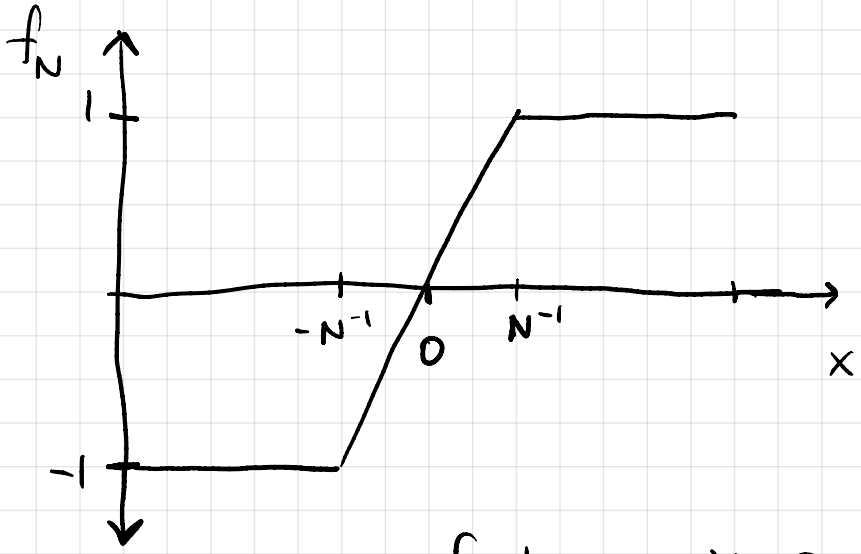
f_N converges pointwise to $\tilde{f}(x) = \begin{cases} 1, & x=0 \\ 0, & x\in(0,1] \end{cases}$

$$\sup_{x\in[0,1]} |e^{-Nx} - \tilde{f}(x)| \geq |e^{-N0^+} - \tilde{f}(0^+)| = 1$$

so f_N does not converge uniformly ~~to~~ to $\tilde{f}$ (it does not have a uniform limit).

Ex

Let $f_N(x) = \begin{cases} -1 & x \leq -N^{-1} \\ Nx & x\in(-N^{-1}, N^{-1}) \\ 1 & x \geq N^{-1} \end{cases}$



Let $f(x) = \begin{cases} -1 & x<0 \\ 0 & x=0 \\ 1 & x>0 \end{cases}$

Then $f_N \xrightarrow{\text{Pointwise}} f$ on $[-1,1]$ since

$f_N(0) = f(0)$, and for any $x\in[-1,1]\setminus\{0\}$

$f_N(x) = f(x)$ when $N \geq \frac{1}{|x|}$.

And $\|f_N - f\|^2 = \int_{-N^{-1}}^{0} (Nx+1)^2 dx + \int_{0}^{N^{-1}} (Nx-1)^2 dx$

$= 2\int_0^{N^{-1}} (Nx-1)^2 dx = 2N^{-1}\int_0^1 (y-1)^2 dy$

$\to 0$ as $N \to \infty$

So $f_N \to f$ in MSS on $[-1,1]$

But $\|f_N - f\|_\infty = \sup_{x\in[-1,1]} |f_N(x) - f(x)| = 1$

for any $N \in \mathbb{N}$, so f_N does not converge uniformly to f.

Properties For $\{f_n\}_{N=1}^{\infty} \subseteq \mathcal{P}$,

Uniform conv $\Rightarrow$ pointwise conv and MSS conv

Proof If $\lim_{N\to\infty} \|f_N - f\|_\infty = 0$ for some $f \in \mathcal{P}$,

then for any $x \in [a,b]$

$\lim_{N\to\infty} |f_N(x) - f(x)| \le \lim_{N\to\infty} \sup_{y\in[a,b]} |f_N(y) - f(y)|$

$= \lim_{N\to\infty} \|f_N - f\|_\infty = 0$

and

$\|f_N - f\|^2 = \int_a^b |f_N(x) - f(x)|^2 dx \le \|f_N - f\|_\infty^2 (b-a)$

$\to 0$ as $N \to \infty$.

∎

7.19 Consider 2D heat eq.

$u_t = \Delta u$ in $\Omega, t \geq 0$

$u(x,y,t) = 0$ for $(x,y) \in \partial\Omega$

$u(x,y,0) = f(x,y)$ $(x,y) \in \Omega$

a) Show that for sufficiently smooth sol,

$$E(t) = \iint_\Omega u^2(x,y,t)\,dx\,dy \leq \iint_\Omega f^2(x,y)\,dx\,dy$$

for $t \geq 0$.

Sol: (for $u \in C^{2,1}(\bar{\Omega} \times [0,\infty))$

$$E'(t) = \iint_\Omega \frac{d}{dt} u^2\,dx\,dy$$

$$= 2\iint_\Omega u_t u\,dx\,dy$$

$$= 2\iint_\Omega \Delta u\, u\,dx\,dy$$

div thm

$$= -2\iint_\Omega |\nabla u|^2\,dx\,dy + \underbrace{\int_{\partial\Omega} u \frac{\partial u}{\partial n}\,ds}_{=0}$$

($u = 0$ on $\partial\Omega$)

$$\leq 0$$

$$\Rightarrow E(t) \leq E(0) = \iint_\Omega f^2(x,y)\,dx\,dy$$

$\forall\ t \geq 0$.

(b) If $u, v \in C^{2,1}(\bar{\Omega} \times [0,\infty))$ both solve PDE, then $w = u - v$ solves

$$w_t = \Delta w \quad \text{in } \Omega \times (0,\infty)$$
$$w = 0 \quad \text{on } \partial\Omega \times [0,\infty)$$
$$w = 0 \quad \text{on } \Omega \times \{0\}$$

By energy arg

$$\iint_\Omega w^2(x,y,t)\,dx\,dy \le \iint_\Omega w^2(x,y,0)\,dx\,dy = 0$$

$\Rightarrow w \equiv 0 \Rightarrow u = v.$

c) For $\Omega = (0,1)$, use Ex 7.17 b) to construct formal sol.

Ansatz $V_{jk}(x,y,t) = T_{jk}(t)\,u_{jk}(x,y)$

leads to $(\partial_t V_{jk} = \Delta V_{jk})$

$$\frac{T_{jk}'(t)}{T_{jk}(t)} = \frac{\Delta u_{jk}}{u_{jk}} = -\lambda_{jk}$$

with $u_{jk}|_{\partial\Omega} = 0$. From 7.(7 b)

$u_{jk} = \sin(j\pi x)\sin(k\pi y)$, $\lambda_{jk} = (j^2+k^2)\pi^2$

$$T_{jk}(t) = e^{-\lambda_{jk} t} = e^{-(j^2+k^2)\pi^2 t}$$

Formal sol

$$u(x,y,t) = \sum_{j,k=1}^{\infty} c_{jk} e^{-(j^2+k^2)\pi^2 t} \sin(j\pi x)\sin(k\pi y)$$

where $u(x,y,0) = f(x,y)$

$$\Rightarrow$$

$$c_{jk} = \frac{\int_0^1\int_0^1 f(x,y)\sin(j\pi x)\sin(k\pi y)\,dx\,dy}{\int_0^1\int_0^1 \sin^2(j\pi x)\sin^2(k\pi y)\,dx\,dy}$$

$$=: \frac{\langle f, u_{jk}\rangle}{\langle u_{jk}, u_{jk}\rangle}$$

$$= 4\langle f, u_{jk}\rangle .$$

Lecture Friday

Question from yesterday:

$f_n(x) = e^{-nx}$ has pointwise limit

$$\tilde{f}(x) = \begin{cases} 1 & x = 0 \\ 0 & x \in (0,1] \end{cases} \quad \text{on } [0,1]$$

Does there exist an $f \in \mathcal{P}$ s.t. $\|f_n - f\|_\infty \to 0$?

A: No. The result $\|f_n - f\|_\infty \to 0 \Rightarrow f_n \xrightarrow{pw} f$, implies in this case that either $f = \tilde{f}$ or f_n does not have uniform limit. And

$$\|f_n - \tilde{f}\|_\infty \geq |f_n(0+) - \tilde{f}(0+)| = 1 \quad \forall n \geq 1.$$

Today

- $f \in \mathcal{P}$ + one-sided differentiable $\Rightarrow$

$$S_N(f)(x) \xrightarrow{\text{pointwise}} \frac{f(x+) + f(x-)}{2} \quad \text{on } [-1,1]$$

- $f' \in \mathcal{P}$ + periodic

$$\Rightarrow S_N(f) \xrightarrow{\text{uniformly}} f \quad \text{on } [-1,1]$$

One-sided differentiable functions

Let $\mathcal{P}^1[-1,1] \subseteq \mathcal{P}[-1,1]$ denote subset of one-sided differentiable functions: For any $f \in \mathcal{P}^1$

$\exists$ a finite number of points $-1 = x_0 < x_1 < \cdots < x_{k+1} = 1$, such that $f \in C^1(x_i, x_{i+1})$ and limits

$f'(x_i^+) = \lim_{x \downarrow x_i} f(x)$, $f'(x_{i+1}^-) = \lim_{x \uparrow x_{i+1}} f'(x)$ exist

for all $i = 0, 1, \ldots, k$.

NB! $f \in \mathcal{P}^1$ does not require that $f' \in \mathcal{P}$, only that $f' : (x_i, x_{i+1}) \to \mathbb{R}$ and limits defined $\forall i$.

($f'(x_i)$ need not be defined)

Shorthand notation: $\mathcal{P}^1 = \mathcal{P}^1[-1,1]$.

Ex • $f(x) = |x|$ belongs to $\mathcal{P}^1$.

$f \in C^1(-1,0)$ and $f \in C^1(0,1)$

with $f'(0\pm) = \pm 1$ and $f'(-1+) = -1$

$f'(1-) = 1$.

• $f(x) = \text{sign}(x) \in \mathcal{P}^1$

$f \in C^1(-1,0)$ and $f \in C^1(0,1)$

$f'(0\pm) = 0$, $f'(-1+) = 0$, $f'(1-) = 0$.

Note: $\text{sign}'(x)$ "$=$" $\begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$

so $\text{sign}'(x) \notin \mathcal{P}$ since it's not a well-defined function in $x = 0$.

Representations of $S_N(f)$.

$$S_N(f)(x) = \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos(k\pi x) + b_k \sin(k\pi x) \quad \in \mathcal{P},$$

with $a_k = \int_{-1}^{1} \cos(k\pi y) f(y)\, dy$

$b_k = \int_{-1}^{1} \sin(k\pi y) f(y)\, dy$

Alternative representation:

$$S_N(f)(x) = \int_{-1}^{1} \left[\frac{1}{2} + \sum_{k=1}^{N} \underbrace{\cos(k\pi y)\cos(k\pi x) + \sin(k\pi y)\sin(k\pi x)}_{\cos(k\pi(x-y))}\right] f(y)\, dy$$

$$= \frac{1}{2}\int_{-1}^{1} \left[1 + 2\sum_{k=1}^{N} \cos(k\pi(x-y))\right] f(y)\, dy$$

$$= \frac{1}{2}\int_{-1}^{1} K_N(x-y) f(y)\, dy, \qquad (1)$$

where $K_N(z) := 1 + 2\sum_{k=1}^{N} \cos(k\pi z)$ is the Dirichlet kernel. Properties:

- $K_N(z)$ is a symmetric, 2-periodic function
- $\frac{1}{2}\int_{-1}^{1} K_N(z)\, dz = \frac{1}{2}\int_{-1}^{1} 1\, dz = 1 \qquad (2)$
- $K_N(z) = \begin{cases} 2N+1, & z = 0, \pm 2, \pm 4, \dots \\ \dfrac{\sin\left((N+\frac{1}{2})\pi z\right)}{\sin\left(\frac{\pi z}{2}\right)}, & \text{otherwise} \end{cases} \qquad (3)$

which can be shown from

$$K_N(z) = 1 + 2\sum_{k=1} \cos(k\pi z) = 1 + \sum_{k=1}^{N} e^{ik\pi z} + e^{-ik\pi z}$$

$$= \sum_{k=-N}^{N} e^{ik\pi z} = \sum_{k=-N}^{N} (e^{i\pi z})^k .$$

↑ Geometric series

Thm 1 (Pointwise convergence)

Let $f \in \mathcal{P}^1[-1,1]$ and let f_{per} be its 2-periodic extension from $[-1,1)$ to $\mathbb{R}$. Then it holds that

$$\lim_{N\to\infty} S_N(f)(x) = \frac{f_{per}(x-) + f_{per}(x+)}{2} \quad \forall x \in [-1,1]$$

Proof For $x \in [0,1]$ (similar arg for $x \in [-1,0)$),

$$S_N(f)(x) - \tfrac{1}{2}\left(f_{per}(x+) + f_{per}(x-)\right)$$

$$\overset{(1),(2)}{=} \frac{1}{2}\int_{-1}^{1} K_N(x-y)\left[f(y) - \tfrac{1}{2}\left(f_{per}(x+) + f_{per}(x-)\right)\right] dy$$

$K_N(z)$ – symmetric & 2-periodic

$z = y - x$

$$= \frac{1}{2}\int_{-1-x}^{1-x} K_N(-z)\left[f(x+z) - \tfrac{1}{2}\left(f_{per}(x+) + f_{per}(x-)\right)\right] dz$$

$$= \frac{1}{2}\int_{-1}^{1-x} K_N(z)\left[f(x+z) - \tfrac{1}{2}\left(f_{per}(x+) + f_{per}(x-)\right)\right] dz$$

$$+\frac{1}{2}\int_{-1-x}^{-1} \underbrace{K_N(z+2)}_{=K_N(z)} \Big[\underbrace{f_{per}(x+z+2)}_{=f(x+z)} - \frac{1}{2}\left(f_{per}(x+) + f_{per}(x-)\right)\Big] dz$$

$z \to z+2 \Rightarrow \int_{-1-x}^{-1} \to \int_{1-x}^{1}$

$$= \frac{1}{2}\int_{-1}^{1} K_N(z)\left[f_{per}(x+z) - \frac{1}{2}\left(f_{per}(x+) + f_{per}(x-)\right)\right] dz$$

$$= \underbrace{\frac{1}{2}\int_{0}^{1} K_N(z)\left[f_{per}(x+z) - f(x+)\right] dz}_{=: I_N^+} + \underbrace{\frac{1}{2}\int_{-1}^{0} K_N(z)\left[f_{per}(x+z) - f(x-)\right] dz}_{=: I_N^-}$$

And

$$I_N^+ \overset{(3)}{=} \int_0^1 \sin\left((N+\tfrac{1}{2})\pi z\right) \underbrace{\frac{f_{per}(x+z) - f(x+)}{\sin(\pi z/2)}}_{= g_+(z)} dz$$

$\to 0$ as $N \to \infty$

Due to following properties:

- $f_{per} \in \mathcal{P}^1[-1,1] \Rightarrow g \in C^1(0, z_1)$ for some $z_1 > 0$, and therefore $\lim_{z \to 0} |g_+(z)| = \frac{2}{\pi}|f'(x+)| < \infty$ (Exercise (see TW P. 293))
- So we can view $g_+ \in C[0, z_1] \Rightarrow g_+ \in \mathcal{P} \Rightarrow \|g_+\| < \infty$
- Sequence $X_j(z) = \sin((j+\frac{1}{2})\pi z)$ is orthogonal

$$\langle X_j, X_k \rangle = \int_0^1 X_j(z) X_k(z)\, dz = 0 \quad j \neq k$$

(Ex 8.4 in TW)

so by Bessel's ineq $\sum_{j=1}^{\infty} \langle X_j, g_+ \rangle^2 \leq \|g_+\|^2$

$\Rightarrow (I_N^+)^2 = \langle X_N, g_+ \rangle^2 \to 0$ as $N \to \infty$.

The limit $I_N^- \xrightarrow{N\to\infty} 0$ is shown similarly

Summary: $S_N(f)(x) - \frac{1}{2}(f_{per}(x+) + f_{per}(x-)) = I_N^+ + I_N^- \xrightarrow{N\to\infty} 0$ ∎

Uniform convergence.

Theorem 2 Let $f' \in \mathcal{P}[-1,1]$ and periodic $f(-1) = f(1)$. Then

$$S_N(f)(x) = \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos(k\pi x) + b_k \sin(k\pi x)$$

converges uniformly to f on $[-1,1]$.

Proof Let f_{per} be 2-periodic extension of f from $[-1,1)$ to $\mathbb{R}$. Then $f_{per} \in C(\mathbb{R})$ since $f' \in \mathcal{P} \Rightarrow f \in C[-1,1]$, and $f(-1) = f(1)$. Moreover, $f' \in \mathcal{P} \Rightarrow f \in \mathcal{P}'$, so Thm 1 implies pointwise convergence

$$\lim_{N\to\infty} S_N(f)(x) = \frac{f_{per}(x+) + f_{per}(x-)}{2} = f(x) \quad \forall x \in [-1,1]$$

Since $S_\infty(f)(x) = f(x)$ $\forall x$ (pointwise conv), we have $f - S_N(f) = S_\infty(f) - S_N(f) = \sum_{k=N+1}^{\infty} a_k \cos(k\pi x) + b_k \sin(k\pi x)$

$$\Rightarrow \|f - S_N(f)\|_\infty = \sup_{x\in[-1,1]} \left| \sum_{k=N+1}^{\infty} a_k \cos(k\pi x) + b_k \sin(k\pi x) \right|$$

$$\leq \sum_{k=N+1}^{\infty} |a_k| + |b_k| \to 0 \text{ as } N\to\infty$$

(remains to show)

Let $\alpha_k = \int_{-1}^{1} f'(y)\cos(k\pi y)\,dy = \begin{cases} 0, & k=0 \\ k\pi b_k, & k\geq 1 \end{cases}$ (4)

$\beta_k = \int_{-1}^{1} f'(y)\sin(k\pi y)\,dy = -k\pi a_k$

Since $f' \in \mathcal{P}$, Bessel's inequality yields

$$\frac{\alpha_0^2}{2} + \sum_{k=1}^{\infty} \alpha_k^2 + \beta_k^2 \leq \int_{-1}^{1} (f'(y))^2\,dy < \infty$$

By (4),

$$\sum_{k=1}^{N} |a_k| + |b_k| = \frac{1}{\pi} \sum_{k=1}^{N} \frac{1}{k} (|\alpha_k| + |\beta_k|)$$

Cauchy-Schwarz in $\mathbb{R}^N$

$$\leq \frac{1}{\pi} \left(\underbrace{\sum_{k=1}^{N} \frac{1}{k^2}}_{\leq \frac{\pi^2}{6}} \right)^{\frac{1}{2}} \left(\sum_{k=1}^{N} \underbrace{(|\alpha_k| + |\beta_k|)^2}_{\leq 2(\alpha_k^2 + \beta_k^2)} \right)^{\frac{1}{2}}$$

$$\leq \sqrt{\frac{2}{6}} \left(\sum_{k=1}^{N} \alpha_k^2 + \beta_k^2 \right)^{\frac{1}{2}}$$

$$\leq \frac{\|f'\|}{\sqrt{3}} < \infty \qquad \forall N \geq 1$$

$$\Rightarrow \sum_{k=1}^{\infty} |a_k| + |b_k| < \infty$$

$$\Rightarrow \sum_{k=N+1}^{\infty} |a_k| + |b_k| \to 0 \text{ as } N\to\infty$$

∎

Example. For $f(x)=|x|$, we have $f\in C[-1,1]$ $f'=\operatorname{sgn}(x)\in \mathcal{P}$ and $f(-1)=f(1)$. Theorem 2 applies: $\lim_{N\to\infty}\|S_N(f)-f\|_\infty=0$.

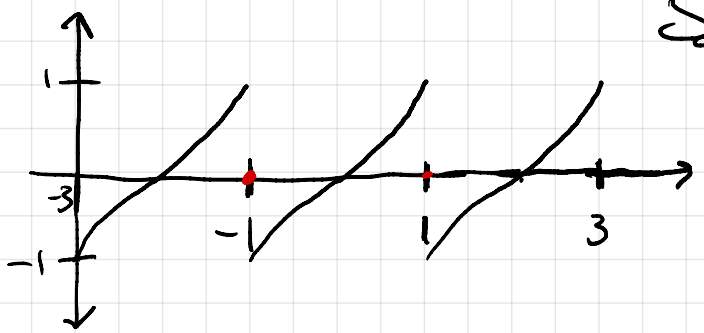
- For $f(x)=x^3$, $f\in C[-1,1]$ and $f'\in\mathcal{P}$, but $f(-1)\neq f(1)$ so Thm 2 does not apply. However $f\in\mathcal{P}'$ so Thm 1 applies: $S_\infty(f)(x)=\dfrac{f_{per}(x-)+f_{per}(x+)}{2}$

Particularly:

$$S_\infty(f)(-1)=\frac{f_{per}(1-)+f_{per}(1+)}{2}=\frac{1+(-1)}{2}=0$$

$S_\infty(f)(1)$
$S_\infty(f)(3)$
$S_\infty(f)(5)$
$\vdots$

$S_\infty(f)(x)$



Uniform convergence on $[-1,1]$ does not hold since $0=S_\infty(f)(\pm1)\neq f(\pm1)=\pm1$, so

$$\lim_{N\to\infty}\|S_N(f)-f\|_\infty\geq\lim_{N\to\infty}|S_N(f)(\pm1)-f(\pm1)|=1\neq0.$$

Ex 8.1, 8.2, 8.3, 8.4a), 8.6

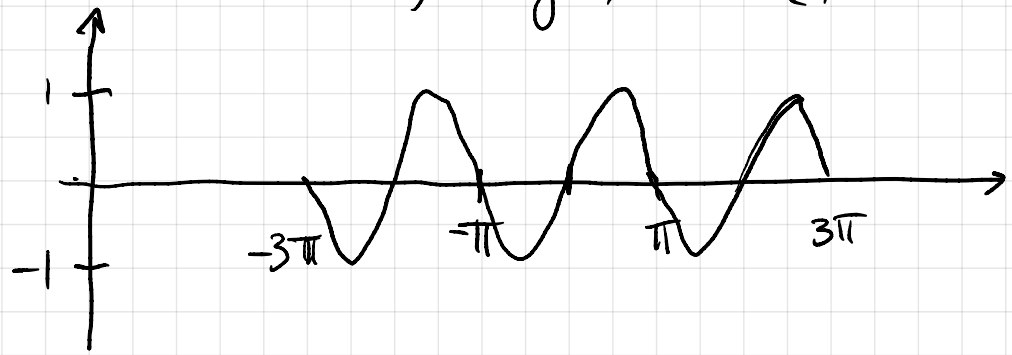
8.1 Sketch periodic extension of

a) $f(x) = |x|$ on $[-1, 1]$ $\quad f_{per}(x+2k) = f(x)$ $x \in [-1,1]$ $\forall k \in \mathbb{Z}$



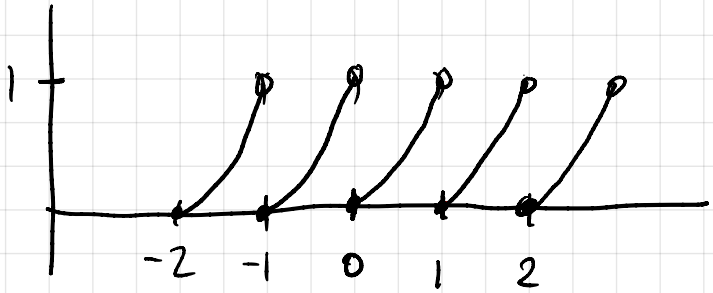
b) $g(x) = \sin(x)$ on $[-\pi, \pi]$ $\quad$ (2π-periodic)

$g_{per}(x + 2k\pi) = g(x) = \sin(x) \quad \forall x \in [-1,1]$ & $k \in \mathbb{Z}$.



c) $h(x) = x^2$ on $[0,1)$ to $\mathbb{R}$

$h_{per}(x+k) = h(x)$ $\quad x \in [0,1)$ & $\forall k \in \mathbb{Z}$



8.2

a) If f, g both even or both odd then $(fg)(x)$ is even

Proof: both even: $(fg)(-x) = f(-x)\,g(-x) = f(x)g(x)$
$= (fg)(x)$.

both odd $(fg)(-x) = f(-x)g(-x)$
$= (-f(x))(-g(x)) = (fg)(x)$

b) If f is even g is odd or vice versa, then $(fg)(x)$ is odd.

Proof f even, g odd:

$$(fg)(-x) = f(-x)\,g(-x) = f(x)(-g(x)) = -(fg)(x).$$

c) If f is odd, then $\int_{-1}^{1} f(x)\,dx = 0$

Sol: $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx$ $[y = -x]$

$$= \int_{0}^{1} \underbrace{f(-y)}_{-f(y)}\,dy + \int_{0}^{1} f(x)\,dx$$

$$= 0$$

8.3 Full Fourier series on $[-1,1]$ of

a) $f(x) = x^2$

$$a_0 = \langle x^2, 1 \rangle = \int_{-1}^{1} x^2 \, dx = \frac{2}{3}$$

$k \geq 1$: $a_k = \langle x^2, \cos(k\pi x) \rangle = \int_{-1}^{1} x^2 \cos(k\pi x) \, dx$

$$= \left[\frac{x^2 \sin(k\pi x)}{k\pi} \right]_{-1}^{1} - \frac{2}{k\pi} \int_{-1}^{1} x \sin(k\pi x) \, dx$$

$$= \frac{2}{(k\pi)^2} \left[x \cos(k\pi x) \right]_{-1}^{1} - \frac{2}{(k\pi)^2} \int_{-1}^{1} \cos(k\pi x) \, dx$$

$$= \frac{4}{(k\pi)^2} (-1)^k + \underbrace{\frac{2}{(k\pi)^3} \left[\sin(k\pi x) \right]_{-1}^{1}}_{=0}$$

$$b_k = \langle x^2, \sin(k\pi x) \rangle = \int_{-1}^{1} \underbrace{x^2 \sin(k\pi x)}_{\text{odd function}} \, dx = 0$$

$$\Rightarrow S_\infty(f)(x) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos(k\pi x)$$

b) $f(x) = e^x$

$$a_0 = \langle e^x, 1 \rangle = \int_{-1}^{1} e^x \, dx = e^1 - e^{-1}$$

$k \geq 1$: $a_k = \int_{-1}^{1} e^x \cos(k\pi x) \, dx$

$$= \underbrace{\frac{1}{k\pi} \left[e^x \sin(k\pi x) \right]_{-1}^{1}}_{=0} - \frac{1}{k\pi} \int_{-1}^{1} e^x \sin(k\pi x) \, dx$$

$$= \frac{1}{(k\pi)^2}\left[e^x \cos(k\pi x)\right]_{-1}^{1} - \frac{1}{(k\pi)^2}\int_{-1}^{1} e^x \cos(k\pi x)\,dx$$

$$= \frac{(-1)^k}{(k\pi)^2}\left(e^1 - e^{-1}\right) - \frac{a_k}{(k\pi)^2}$$

$$\Rightarrow a_k\left(1 + \frac{1}{(k\pi)^2}\right) = \frac{(-1)^k}{(k\pi)^2}\left(e^1 - e^{-1}\right)$$

$$\Rightarrow a_k = \frac{(-1)^k\left(e^1 - e^{-1}\right)}{\left((k\pi)^2 + 1\right)}$$

$$b_k = \int_{-1}^{1} e^x \sin(k\pi x)\,dx$$

$$= -\frac{1}{(k\pi)}\left[e^x \cos(k\pi x)\right]_{-1}^{1} + \frac{1}{k\pi}\int_{-1}^{1} e^x \cos(k\pi x)\,dx$$

$$= \frac{(-1)^k}{k\pi}\left(e^{-1} - e^1\right) + \frac{(-1)^k}{k\pi}\,a_k$$

$$= (-1)^k\left(e^1 - e^{-1}\right)\left(\frac{1}{((k\pi)^2+1)k\pi} - \frac{1}{k\pi}\right)$$

$$= (-1)^k\left(e^1 - e^{-1}\right)\cdot\frac{1-(1+(k\pi)^2)}{((k\pi)^2+1)\,k\pi}$$

$$= (-1)^{k+1}\left(e^1 - e^{-1}\right)\cdot\frac{k\pi}{1+(k\pi)^2}$$

$$S_\infty(t) = \left(e^1 - e^{-1}\right)/2$$

$$+ \sum_{k=1}^{\infty} \frac{(-1)^k\left(e^1 - e^{-1}\right)}{1+(k\pi)^2}\left(\cos(k\pi x) - k\pi\sin(k\pi x)\right)$$

8.4 a) Find full Fourier series of $f(x) = \sin^2(x)$ on $[-\pi, \pi]$

Comment:

Fourier series on $[-L, L]$ defined by

$$S_\infty(f) = \frac{\hat{a}_0}{2} + \sum_{k=1}^{\infty} \hat{a}_k \cos\left(k\pi \frac{x}{L}\right) + \hat{b}_k \sin\left(k\pi \frac{x}{L}\right)$$

where $\hat{a}_k := \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(k\pi \frac{x}{L}\right) dx$

$\hat{b}_k := \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(k\pi \frac{x}{L}\right) dx$

Can be derived by $\tilde{f}(x) := f(Lx)$

Then find Fourier series of $\tilde{f}$ on $[-1, 1]$.

$S_\infty(\tilde{f})(x) =$

Then series of f on $[-L, L]$ must satisfy

$$S_\infty(f)(x) = S_\infty(\tilde{f})\left(\frac{x}{L}\right)$$

since $f(x) = \tilde{f}\left(\frac{x}{L}\right) \quad \forall \quad x \in [-L, L]$.

(and show that $\hat{a}_k = a_k$, $\hat{b}_k = b_k$ $k \geq 1$)

Solution: $\sin^2(x) + \cos^2(x) = 1$

$$\sin^2(x) - \cos^2(x) = -\cos(2x)$$

$$\Rightarrow \sin^2(x) = \frac{1 - \cos(2x)}{2}$$

$\Rightarrow$ Solution $\hat{a}_0 = 1, \hat{a}_2 = \frac{1}{2}$

$$\mathcal{F}(f)(x) = \frac{\hat{a}_0}{2} + \hat{a}_2 \cos\left(2\pi \frac{x}{\pi}\right)$$

But let us verify by calculation:

$$\hat{a}_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1-\cos(2x)}{2}\,dx$$

$$= 1 + \left[\sin(2x)\right]_{-\pi}^{\pi} = 1$$

$k \geq 1$: $\hat{a}_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\left(k\pi \frac{x}{\pi}\right)\frac{(1-\cos(2x))}{2}\,dx$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(kx) - \cos(kx)\cos(2x)\,dx$$

$$= \underbrace{\frac{1}{k\pi}\left[\sin(k\pi)\right]_{-\pi}^{\pi}}_{=0} - \frac{1}{4\pi}\int_{-\pi}^{\pi} \cos((k+2)x) - \cos((k-2)x)\,dx$$

$$= \begin{cases} 1/2 & k = 2 \\ 0 & k \neq 2 \end{cases}$$

$$\hat{b}_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\overbrace{(1-\cos(2x))}^{\text{even}}\overbrace{\sin(k\pi x)}^{\text{odd}}}_{\text{odd}}\,dx$$

$$= 0.$$

Ex 8.6 $f(x) = \frac{1}{2} - x$

a) Sine series on $[0,1]$

$$B_k = 2\int_0^1 \left(\frac{1}{2} - x\right) \sin(k\pi x)\, dx$$

$$= \frac{-1}{k\pi}\left[(1-2x)\cos(k\pi x)\right]_0^1$$

$$- \frac{2}{k\pi}\int_0^1 \cos(k\pi x)\, dx$$

$$= \frac{1+(-1)^k}{k\pi} \cdot \frac{1}{2} - x \sim \sum_{k=1}^{\infty} \frac{1+(-1)^k}{k\pi} \sin(k\pi x)$$

b) Cosine series on $[0,1]$

$$a_0 = \int_0^1 \left(\frac{1}{2} - x\right) \cdot 1\, dx = 0$$

$k \geq 1$ $\quad a_k = 2\int_0^1 \left(\frac{1}{2} - x\right)\cos(k\pi x)\, dx$

$$= \frac{1}{k\pi}\left[(1-2x)\sin(k\pi x)\right]_0^1$$

$$+ \frac{2}{k\pi}\int_0^1 \sin(k\pi x)\, dx$$

$$= \frac{-2}{(k\pi)^2}\left[\cos(k\pi x)\right]_0^1 = \frac{2}{(k\pi)^2}\left(1-(-1)^k\right)$$

$$\left(\frac{1}{2} - x \sim \sum_{k=1}^{\infty} \frac{2(1+(-1)^{k+1})}{(k\pi)^2} \cos(k\pi x)\right)$$

c) Find Fourier sine series $[0,1]$ of $g(x) = \frac{1}{2}(1-x)x$ using b) above.

Sol: $g'(x) = \frac{1}{2} - x = f$

For $f_{bo}(g) = \sum_{k=1}^{\infty} b_k \sin(k\pi x)$

we obtain relations to b):

$$\alpha_k = \int_0^1 g'(x)\cos(k\pi x)\,dx$$

$$= \left[g(x)\cos(k\pi x)\right]_0^1 + k\pi \int_0^1 g(x)\sin(k\pi x)\,dx$$

$$= (k\pi)\, b_k \quad \text{for } k \geq 1 \text{ \&}$$

$$\Rightarrow b_k = \frac{\alpha_k}{k\pi} = \frac{2}{(k\pi)^3}\left(1-(-1)^k\right)$$