Mean square sense (MSS) convergence

Theorem 1 (Recall from last week)
If $f' \in \mathcal{P}[-1,1]$ and $f(-1) = f(1)$, then
$\lim_{N\to\infty} \|S_N(f) - f\|_\infty = 0$ (uniform conv.)

and

$$\underbrace{\lim_{N\to\infty} \|S_N(f) - f\|_\infty = 0}_{\text{uniform conv}} \Rightarrow \underbrace{\lim_{N\to\infty} \|S_N(f) - f\|_{\text{MSS}} = 0}_{\text{MSS conv}}$$

Theorem 2 (conv in MSS)
If $f \in \mathcal{P}[-1,1]$, then $\lim_{N\to\infty} \|S_N(f) - f\| = 0$

Sketch of proof:

Let $f_{per}(x)$ be 2-periodic extension of f from $[-1,1)$ to $\mathbb{R}$, and for any $\delta \in (0,1)$, let

$$f_\delta(x) := \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} f_{per}(y)\, dy \qquad x \in [-1,1]$$

Properties:

- $\|f - f_\delta\|^2 = \int_{-1}^{1} (f(x) - f_\delta(x))^2 dx \to 0$ as $\delta \to 0$ (1)

(Exercise 9.17).

- f_δ is 2-periodic ($z = y - 2$)

$$\text{as } f_\delta(x+2) = \frac{1}{2\delta} \int_{x+2-\delta}^{x+2+\delta} f_{per}(y)\, dy = \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} f_{per}(z+2)\, dz$$

- $f_\delta'(x) = \dfrac{f_{per}((x+\delta)-) - f_{per}((x-\delta)+)}{2\delta} \in \mathcal{P}$ for any $\delta > 0$,

since both $f_{per}((\cdot+\delta)-)$ and $f_{per}((\cdot-\delta)-)$ belong to $\mathcal{P}$.

Summary: $f_\delta' \in \mathcal{P}$ & $f_\delta(-1) = f_\delta(1)$ for any $\delta > 0$.

So Theorem 1 applies to $f_\delta \Rightarrow$ Conv. in MSS:

$$\lim_{N\to\infty} \|S_N(f_\delta) - f_\delta\| = 0 \quad \text{for any } \delta > 0 \qquad (2)$$

By linearity, $S_N(f) - S_N(f_\delta) = S_N(f - f_\delta)$,
and by Bessel's ineq

$$\|S_N(f-f_\delta)\|^2 \le \|f - f_\delta\|^2.$$

Hence

$$\begin{aligned}\|S_N(f) - f\| &= \|S_N(f-f_\delta) + (S_N(f_\delta) - f_\delta) + (f_\delta - f)\| \\ &\le \|S_N(f-f_\delta)\| + \|f - f_\delta\| + \|S_N(f_\delta) - f_\delta\| \\ &\le 2\|f_\delta - f\| + \|S_N(f_\delta) - f_\delta\|\end{aligned}$$

$$\Rightarrow \lim_{N\to\infty} \|S_N(f) - f\| \le 2\|f_\delta - f\| + \underbrace{\lim_{N\to\infty} \|S_N(f_\delta) - f_\delta\|}_{\overset{(2)}{=} 0}$$

$$\lim_{N\to\infty} \|S_N(f) - f\| = \lim_{\delta\downarrow 0} \lim_{N\to\infty} \|S_N(f) - f\|$$

$$\le \lim_{\delta\downarrow 0} 2\|f_\delta - f\| \overset{(1)}{=} 0$$

∎

Recall

$S_N(t) \to f$ in MSS $\Leftrightarrow$ **Parseval's identity**

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty} a_k^2 + b_k^2 = \|f\|^2 \quad (3)$$

Corollary 1

If $f \in \mathcal{P}[-1,1]$, then Parseval's id (3) holds.

Pf: $f \in \mathcal{P} \Rightarrow S_N(t) \xrightarrow{\text{MSS}} f \Rightarrow (3)$

Example $f(x) = x$ on $[0,1]$ has sine series

$$S_\infty(f)(x) = \sum_{k=1}^{\infty} \underbrace{2\frac{(-1)^{k+1}}{k\pi}}_{= b_k} \sin(k\pi x)$$

Odd extension of f: $\tilde{f}(x) = x$ with Fourier series:

$$S_\infty(\tilde{f})(x) = \sum_{k=1}^{\infty} 2\frac{(-1)^{k+1}}{k\pi} \sin(k\pi x) \quad \text{on } [-1,1]$$

Since $\tilde{f} \in C[-1,1]$, Corollary 1 applies:

$$\sum_{k=1}^{\infty} b_k^2 = \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} = \|\tilde{f}\|^2 = \int_{-1}^{1} x^2 dx = \frac{2}{3}.$$

So

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

Ex For $f(x) = \text{sign}(x)$, $f \in P[-1,1]$, &

$$S_\infty(f) = \sum_{k=1}^{\infty} \underbrace{\frac{4}{\pi(2k-1)}}_{b_{2k-1}} \sin((2k-1)\pi x) \quad \text{on } [-1,1]$$

Corollary 1 $\Rightarrow$

$$\frac{16}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \int_{-1}^{1} (\text{sign}(x))^2 \, dx = 2$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8} \, .$$

Regularity of f and decay of $\{a_k\}, \{b_k\}$

Corollary 1 says that if $f \in P$, then

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty} a_k^2 + b_k^2 = \|f\|^2 < \infty.$$

This implies that $a_k, b_k \to 0$ as $k \to \infty$

(Will show: If f is smooth and periodic, then a_k, b_k decays faster)

Let

$$C_p^m[-1,1] = \{f \in C^m([-1,1]) \mid f^{(j)}(-1) = f^{(j)}(1) \quad j = 0, 1, \ldots, m\}$$

Recall: (Thm 8.1 in TW)

If $f \in C_p^0$ and $f' \in P$, then

$$S_\infty(f') = \sum_{k=1}^{\infty} \alpha_k \cos(k\pi x) + \beta_k \sin(k\pi x)$$

with $\alpha_k = k\pi b_k$, $\beta_k = -k\pi a_k$

where $a_k = \langle f, \cos(k\pi x) \rangle$

$b_k = \langle f, \sin(k\pi x) \rangle$

So if $f \in C_p^0$ & $f' \in \mathcal{P}$, then by Cor 1

$$\sum_{k=1}^{\infty} \alpha_k^2 + \beta_k^2 = \|f'\|^2 < \infty$$

$$\Rightarrow \sum_{k=1}^{\infty} k^2 (a_k^2 + b_k^2) = \frac{\|f'\|^2}{\pi^2} \quad (4)$$

$$\left(\Rightarrow \quad k^2 (a_k^2 + b_k^2) \to 0 \text{ as } k \to \infty. \right)$$

Theorem 3 Let $f \in C_p^{m-1}$ and $f^{(m)} \in \mathcal{P}$ for an integer $m \geq 1$. Then

$$\sum_{k=1}^{\infty} k^{2m} (a_k^2 + b_k^2) = \pi^{-2m} \|f^{(m)}\|^2 \quad (*)$$

where a_k, b_k are Fourier coeffs of f.

Proof Have proved $(*)$ for $m=1$ in (4).

Assume $(*)$ holds for some $m \geq 1$, and that $f \in C_p^m$ & $f^{(m+1)} \in \mathcal{P}$.

Then $f' \in C_p^{m-1}$ & $\frac{d^m}{dx^m} f' = f^{(m+1)} \in \mathcal{P}$, so $(*)$ holds for m implies for f' that

$$\sum_{k=1}^{\infty} k^{2m} (\alpha_k^2 + \beta_k^2) = \frac{\|\frac{d^m}{dx^m} f'\|^2}{\pi^{2m}} = \frac{\|f^{(m+1)}\|^2}{\pi^{2m}}$$

where α_k, β_k are Fourier coeffs of f', with

$\alpha_k = \pi k b_k$ and $\beta_k = -\pi k a_k$.

$$\Rightarrow \sum_{k=1}^{\infty} k^{2(m+1)}(a_k^2+b_k^2) = \frac{\|f^{(m+1)}\|^2}{\pi^{2(m+1)}}.$$

So (*) true for m and $f \in C_p^m$, $f^{(m+1)} \in \mathcal{P}$

$\Rightarrow$ (*) true for $m+1$. Result holds by induction ☒

Example For $f(x) = \text{sign}(x)$,

$$S_\infty(f) = \sum_{k=1}^{\infty} \underbrace{\frac{4}{\pi(2k-1)}}_{b_{2k-1}} \sin((2k-1)\pi x)$$

As $f \in \mathcal{P}$, Cor 1 $\Rightarrow \sum_{k=1}^{\infty} b_k^2 = \|f\|^2 < \infty$,

while $\sum_{k=1}^{\infty} k^2 b_k^2 = \sum_{k=1}^{\infty} \left(\frac{4}{\pi}\right)^2 = \infty$.

Does not contradict Theorem 3 as neither $f \in C_p^0$ nor $f' \in \mathcal{P}$ hold.

Example $f(x) = |x|$ satisfies $f \in C_p^0$ & $f' = \text{sign}(x) \in \mathcal{P}$, and

$$S_\infty(f) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k=1}^{\infty} \left(\frac{1}{2k-1}\right)^2 \cos((2k-1)\pi x).$$

Theorem 3 $\Rightarrow$

$$\sum_{k=1}^{\infty} k^2 a_k^2 = \frac{16}{\pi^4} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\|f'\|^2}{\pi^2} = \frac{2}{\pi^2}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$$ (as we know from earlier example).

8.3 Full Fourier series on $[-1,1]$ of

a) $f(x) = x^2$

$$a_0 = \langle x^2, 1 \rangle = \int_{-1}^{1} x^2 \, dx = \frac{2}{3}$$

$k \geq 1$: $a_k = \langle x^2, \cos(k\pi x) \rangle = \int_{-1}^{1} x^2 \cos(k\pi x) \, dx$

$$= \left[\frac{x^2 \sin(k\pi x)}{k\pi} \right]_{-1}^{1} - \frac{2}{k\pi} \int_{-1}^{1} x \sin(k\pi x) \, dx$$

$$= \frac{2}{(k\pi)^2} \left[x \cos(k\pi x) \right]_{-1}^{1} - \frac{2}{(k\pi)^2} \int_{-1}^{1} \cos(k\pi x) \, dx$$

$$= \frac{4}{(k\pi)^2} (-1)^k + \underbrace{\frac{2}{(k\pi)^3} \left[\sin(k\pi x) \right]_{-1}^{1}}_{=0}$$

$$b_k = \langle x^2, \sin(k\pi x) \rangle = \int_{-1}^{1} \underbrace{x^2 \sin(k\pi x)}_{\text{odd function}} \, dx = 0$$

$$\Rightarrow S_\infty(f)(x) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos(k\pi x)$$

b) $f(x) = e^x$

$$a_0 = \langle e^x, 1 \rangle = \int_{-1}^{1} e^x \, dx = e^1 - e^{-1}$$

$k \geq 1$: $a_k = \int_{-1}^{1} e^x \cos(k\pi x) \, dx$

$$= \underbrace{\frac{1}{k\pi} \left[e^x \sin(k\pi x) \right]_{-1}^{1}}_{=0} - \frac{1}{k\pi} \overbrace{\int_{-1}^{1} e^x \sin(k\pi x) \, dx}^{=b_k} \quad (*)$$

$$= \frac{1}{(k\pi)^2}\left[e^x \cos(k\pi x)\right]_{-1}^{1} - \frac{1}{(k\pi)^2}\int_{-1}^{1} e^x \cos(k\pi x)\,dx$$

$$= \frac{(-1)^k}{(k\pi)^2}\left(e^1 - e^{-1}\right) - \frac{a_k}{(k\pi)^2}$$

$$\Rightarrow a_k\left(1 + \frac{1}{(k\pi)^2}\right) = \frac{(-1)^k}{(k\pi)^2}\left(e^1 - e^{-1}\right)$$

$$\Rightarrow a_k = \frac{(-1)^k\left(e^1 - e^{-1}\right)}{\left((k\pi)^2 + 1\right)}$$

$$b_k \overset{(*)}{=} -k\pi a_k$$

$$= (-1)^{k+1}\left(e^1 - e^{-1}\right)\cdot\frac{k\pi}{1+(k\pi)^2}$$

$$S_\infty(t) = \left(e^1 - e^{-1}\right)/2 + \sum_{k=1}^{\infty} \frac{(-1)^k\left(e^1 - e^{-1}\right)}{1+(k\pi)^2}\left(\cos(k\pi x) - k\pi \sin(k\pi x)\right)$$

8.4 a) Find full Fourier series of $f(x) = \sin^2(x)$ on $[-\pi, \pi]$

Comment:

Fourier series on $[-L, L]$ defined by

$$S_\infty(f) = \frac{\hat{a}_0}{2} + \sum_{k=1}^{\infty} \hat{a}_k \cos\left(k\pi \frac{x}{L}\right) + \hat{b}_k \sin\left(k\pi \frac{x}{L}\right)$$

where $\hat{a}_k := \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(k\pi \frac{x}{L}\right) dx$

$\hat{b}_k := \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(k\pi \frac{x}{L}\right) dx$

Can be derived by $\tilde{f}(x) := f(Lx)$

Then find Fourier series of $\tilde{f}$ on $[-1, 1]$.

$$S_\infty(\tilde{f})(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\pi x) + b_k \sin(k\pi x)$$

Then series of f on $[-L, L]$ must satisfy

$$S_\infty(f)(x) = S_\infty(\tilde{f})\left(\frac{x}{L}\right)$$

since $f(x) = \tilde{f}\left(\frac{x}{L}\right) \quad \forall \quad x \in [-L, L]$.

(and show that $\hat{a}_k = a_k$, $\hat{b}_k = b_k$ $k \geq 1$)

Solution: Seek $S_\infty(f) = \frac{\hat{a}_0}{2} + \sum_{k=1}^{\infty} \hat{a}_k \cos(kx) + \hat{b}_k \sin(kx)$

$\sin^2(x) + \cos^2(x) = 1$

$\sin^2(x) - \cos^2(x) = -\cos(2x)$

$\Rightarrow \sin^2(x) = \frac{1 - \cos(2x)}{2}$

$\Rightarrow$ Solution $\hat{a}_0 = 1, \hat{a}_2 = \frac{-1}{2}$

$$\int(t)(x) = \frac{\hat{a}_0}{2} - \hat{a}_2 \cos\left(\frac{2\pi x}{\pi}\right)$$

But let us verify by calculation:

$$\hat{a}_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1-\cos(2x)}{2}\,dx$$

$$= 1 + \left[\sin(2x)\right]_{-\pi}^{\pi} = 1$$

$k \geq 1$: $\hat{a}_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\left(k\pi \frac{x}{\pi}\right)\frac{(1-\cos(2x))}{2}\,dx$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(kx) - \cos(kx)\cos(2x)\,dx$$

$$= \underbrace{\frac{1}{k\pi}\left[\sin(k\pi)\right]_{-\pi}^{\pi}}_{=0} - \frac{1}{4\pi}\int_{-\pi}^{\pi} \cos((k+2)x) + \cos((k-2)x)\,dx$$

$$= \begin{cases} -1/2 & k = 2 \\ 0 & k \neq 2 \end{cases}$$

$$\hat{b}_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\overbrace{(1-\cos(2x))}^{\text{even}}\overbrace{\sin(k\pi x)}^{\text{odd}}}_{\text{odd}}\,dx$$

$$= 0.$$

Ex 8.6 $f(x) = \frac{1}{2} - x$

a) Sine series on $[0,1]$

$$B_k = 2\int_0^1 \left(\frac{1}{2} - x\right) \sin(k\pi x)\, dx$$

$$= \frac{-1}{k\pi}\left[(1-2x)\cos(k\pi x)\right]_0^1$$

$$- \frac{2}{k\pi}\int_0^1 \cos(k\pi x)\, dx$$

$$= \frac{1+(-1)^k}{k\pi} \cdot \left(\frac{1}{2} - x \sim \sum_{k=1}^{\infty} \frac{1+(-1)^k}{k\pi} \sin(k\pi x)\right)$$

b) Cosine series on $[0,1]$

$$a_0 = \int_0^1 \left(\frac{1}{2} - x\right) \cdot 1\, dx = 0$$

$k \geq 1$ $\quad a_k = 2\int_0^1 \left(\frac{1}{2} - x\right)\cos(k\pi x)\, dx$

$$= \frac{1}{k\pi}\left[(1-2x)\sin(k\pi x)\right]_0^1$$

$$+ \frac{2}{k\pi}\int_0^1 \sin(k\pi x)\, dx$$

$$= \frac{-2}{(k\pi)^2}\left[\cos(k\pi x)\right]_0^1 = \frac{2}{(k\pi)^2}\left(1-(-1)^k\right)$$

$$\left(\frac{1}{2} - x \sim \sum_{k=1}^{\infty} \frac{2(1+(-1)^{k+1})}{(k\pi)^2} \cos(k\pi x)\right)$$

c) Find Fourier sine series $[0,1]$ of $g(x) = \frac{1}{2}(1-x)x$ using a), b) above.

Sol: $g'(x) = \frac{1}{2} - x = f$

For $f_{bo}(g) = \sum_{k=1}^{\infty} b_k \sin(k\pi x)$

we obtain relation to b):

$$\alpha_k = \int_0^1 g'(x)\cos(k\pi x)\,dx$$

$$= \left[g(x)\cos(k\pi x)\right]_0^1 + k\pi \int_0^1 g(x)\sin(k\pi x)\,dx$$

$$= (k\pi)\, b_k \quad \text{for } k \geq 1 \text{ \&}$$

$$\Rightarrow b_k = \frac{\alpha_k}{k\pi} = \frac{2}{(k\pi)^3}\left(1-(-1)^k\right)$$

Lecture Friday

Lemma 1 If $\{f_N\}_{N=1}^{\infty} \subset C[a,b]$ and $\exists f:[a,b]\to\mathbb{R}$ s.t. $\|f-f_N\|_\infty \to 0$ as $N\to\infty$ on $[a,b]$, then $f\in C[a,b]$. (so $C[a,b]$, $\|\cdot\|_\infty$ is closed space.)

Proof Given $\epsilon>0$, let $N_0\in\mathbb{N}$ be such that $\|f_{N_0}-f\|_\infty < \frac{\epsilon}{3}$. Since $f_{N_0}\in C[a,b]$, let $\delta>0$ be such that

$$|f_{N_0}(x)-f_{N_0}(y)|<\tfrac{\epsilon}{3} \quad \text{whenever} \quad |x-y|<\delta$$

Hence, for any $y\in B_\delta(x)$,

$$|f(x)-f(y)| \le |f(x)-f_{N_0}(x)|+|f_{N_0}(x)-f_{N_0}(y)|+|f_{N_0}(y)-f(y)|$$

$$\le 2\|f-f_{N_0}\|_\infty + \tfrac{\epsilon}{3} \le \epsilon.$$

So for any $\epsilon>0$, we can find $\delta>0$ s.t.

$$|f(x)-f(y)|<\epsilon \quad \text{whenever} \quad |x-y|<\delta$$

$$\Rightarrow f\in C[a,b]$$

∎

Lemma 2 Let $\{f_N\} \subset C^1[a,b]$ and let there be functions $f, g \in C[a,b]$ s.t.

$\|f - f_N\|_\infty \to 0$ and $\|g - f_N'\|_\infty \to 0$ as $N \to \infty$.

Then $f \in C^1[a,b]$ and $f' = g$.

(so $C^1[a,b], (\|\cdot\|_\infty + \|\cdot'\|_\infty)$ is closed space)

Proof Let $\tilde{f}(x) := f(a) + \int_a^x g(y)\,dy$

where $f(a)$ is value in $x=a$ of the limit function f. So value $f(a)$ exists by assumption, and we note for later that

$$\lim_{N\to\infty} |f_N(a) - f(a)| \le \lim_{N\to\infty} \|f - f\|_\infty = 0.$$

and that $\tilde{f} \in C^1[a,b]$ since $g \in C[a,b]$.

Let us next verify that $\tilde{f} = f$.

By fund. thm of calc,

$$f_N(x) = f_N(a) + \int_a^x f_N'(y)\,dy$$

and

$$|\bar{f}(x) - f_N(x)| \le |f(a) - f_N(a)| + \int_a^x |g(y) - f_N'(y)|\, dy$$

$$\le \|f - f_N\|_\infty + (x-a)\|g - f_N'\|_\infty$$

$$\Rightarrow \sup_{x\in[a,b]} |\bar{f}(x) - f_N(x)| \le \|f - f_N\|_\infty + (b-a)\|g - f_N'\|_\infty$$

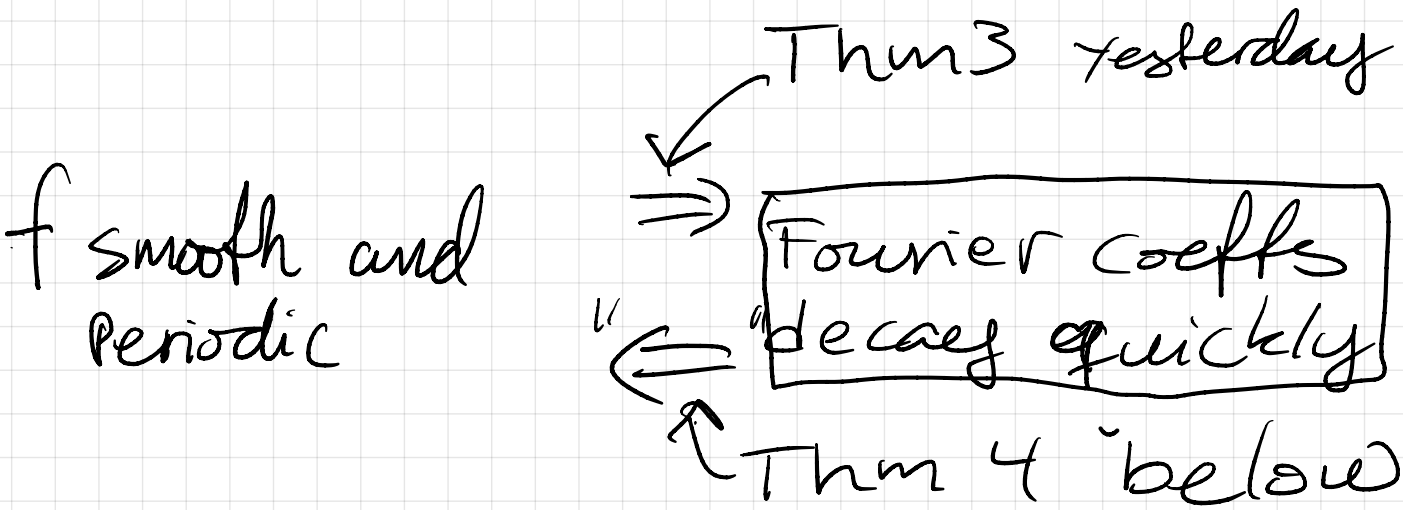
$$\to 0 \text{ as } N \to \infty.$$

Hence $f_N \to \bar{f}$ uniformly on $[-1, 1]$, and since the limit is unique,

$$f = \lim_{N\to\infty} f_N = \bar{f}.$$

Moreover $f'(x) = \bar{f}'(x) = g(x)$.

□



Theorem 4 Let $m \geq 1$ be an integer.

If $P_N(x) := \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos(k\pi x) + b_k \sin(k\pi x)$

with $\sum_{k=1}^{\infty} k^{2m}(a_k^2 + b_k^2) < \infty$ (1),

then $\exists\, f \in C_p^{m-1}$ (possibly not with $f^{(m)} \in P$)

s.t. $P_N^{(j)} \to f^{(j)}$ for all $0 \leq j \leq m-1$ (2)

uniformly as $N \to \infty$. And $S_\infty(t) = P_\infty$.

Proof Two parts: ① for $m=1$

② induction for $m > 1$.

① $m=1$. For each $x \in [-1,1]$, $\{P_N(x)\}_{N=1}^{\infty}$ is a Cauchy seq in $\mathbb{R}$ since for any $M > N \geq 1$,

$$|P_M(x) - P_N(x)| \leq \sum_{k=N+1}^{M} |a_k| + |b_k| = \sum_{k=N+1}^{M} k^{-1}(k(|a_k| + |b_k|))$$

$$\leq \underbrace{\left(\sum_{k=N+1}^{M} \frac{1}{k^2}\right)^{\frac{1}{2}}}_{\to 0} \underbrace{\left(\sum_{k=N+1}^{M} 2k^2(a_k^2 + b_k^2)\right)^{\frac{1}{2}}}_{\to 0}$$

both as $M, N \to \infty$.

$\Rightarrow$ pointwise convergence

$$\lim_{N\to\infty} P_N(x) =: \underbrace{f(x)}_{\text{we call limit } f(x)} \qquad \forall\, x \in [-1, 1]$$

Convergence is also uniform:

$$\|f - P_N\|_\infty = \|P_\infty - P_N\|_\infty \leq \sum_{k=N+1}^{\infty} |a_k| + |b_k|$$

again

$$\leq \underbrace{\left(\sum_{k=N+1}^{\infty} \frac{1}{k^2}\right)^{\frac{1}{2}}}_{\to 0} \underbrace{\left(\sum_{k=N+1}^{\infty} 2k^2(a_k^2 + b_k^2)\right)}_{< \infty}$$

$\to 0$ as $N \to \infty$.

Since $P_N \in C_P^0$ for each $N \geq 1$, Lemma 1 $\Rightarrow f \in C[-1,1]$. Moreover ($P_N$ 2-periodic $\forall N$)

$$f(1) = \lim_{N\to\infty} P_N(1) = \lim_{N\to\infty} P_N(-1) = f(-1) \Rightarrow f \in C_P^0.$$

Moreover, (Exercise: why?)

$$\left.\begin{aligned} \langle f, \cos(k\pi x)\rangle &= \lim_{N\to\infty} \langle P_N, \cos(k\pi x)\rangle = a_k \\ \langle f, \sin(k\pi x)\rangle &= \lim_{N\to\infty} \langle P_N, \sin(k\pi x)\rangle = b_k \end{aligned}\right\} \Rightarrow S_\infty(f) = P_\infty$$

Part ②: Assume statement Θ:

"(1) ⇒ (2)" is true for some $m \geq 1$,

and that $\sum_{k=1}^{\infty} k^{2(m+1)}(a_k^2 + b_k^2) < \infty$. (3)

Then for

$$Q_N(x) := P_N'(x) = \sum_{k=1}^{N} \alpha_k \cos(k\pi x) + \beta_k \sin(k\pi x) \quad (*)$$

with $\alpha_k = k\pi b_k$, $\beta_k = -k\pi a_k$ we obtain

$$\sum_{k=1}^{\infty} k^{2m}(\alpha_k^2 + \beta_k^2) = \pi^2 \sum_{k=1}^{\infty} k^{2(m+1)}(a_k^2 + b_k^2) \overset{(3)}{<} \infty,$$

So property (A) for m holds for both P_N & Q_N, and statement Θ: (1) ⇒ (2) true for m means when

- applied to P_N that for some $f \in C_P^m$, $P_N^{(j)} \to f^{(j)}$ uniformly for $0 \leq j \leq m-1$
- applied to Q_N that for some $g \in C_P^m$, $Q_N^{(j)} \to g^{(j)}$ uniformly for all $0 \leq j \leq m-1$.

Since $Q_N = P_N'$, we have that $Q_N^{(j)} = P_N^{(j+1)}$, so $g^{(j)} = f^{(j+1)}$ $0 \leq j \leq m-2$

Lastly, $g^{(m-1)} = f^{(m)}$ follows by Lemma 2 as $P_N^{(m)} = Q_N^{(m-1)} \to g^{(m-1)}$ & $P_N^{(m-1)} \to f^{(m-2)}$.

Summary: If Statement Θ true for m and (3) holds, then Θ is also true for $m+1$.
If (3) also holds for $m+2$, iterate.
Proof holds by induction.



8.10 Eigenval problem

$$-X''(x) = \lambda X(x) \qquad 0<x<1$$

$$X'(0) = 2X(0), \quad X'(1) = X(1) \quad \text{Robin BC}$$

a) If (λ, X) is eigenpair, then show following:

$$\lambda \int_0^1 X^2 dx = \int_0^1 -X'' X \, dx$$

$$= \left[-X'X\right]_0^1 + \int_0^1 (X')^2 \, dx$$

$$\overset{BC}{=} 2X^2(0) - X^2(1) + \int_0^1 (X')^2 dx$$

Q: Does equality imply $\lambda > 0$ for all eigenvalues?

A: No, perhaps $\exists$ eigenvector with $X^2(1) \geq 2X^2(0) + \int_0^1 (X')^2 dx$.

b) Show that $\lambda = -\mu^2 < 0$ is neg. eigv. iff $\mu > 0$ satisfies

$$\tanh(\mu) = \frac{\mu}{2-\mu^2} \qquad \left(\tanh(\mu) = \frac{e^{\mu} - e^{-\mu}}{e^{\mu} + e^{-\mu}}\right)$$

Sol $X'' = \mu^2 X$ has general solution

$$X(x) = a e^{\mu x} + b e^{-\mu x}$$

$$X'(x) = \mu(ae^{\mu x} - be^{-\mu x})$$

BC: $X'(0) = 2X(0)$: $a\mu - b\mu = 2a + 2b$

$X'(1) = X(1)$: $a\mu e^{\mu} - b\mu e^{-\mu} = ae^{\mu} + be^{-\mu}$

$\Rightarrow b = a\dfrac{\mu-2}{\mu+2}$ & $a \neq 0$ (as otherwise $a = b = 0 \Rightarrow X \equiv 0$).

& (Inserted into second eq.)

$$a\left(\mu e^{\mu} - \frac{(\mu^2 - 2\mu)}{\mu+2}e^{-\mu}\right) = a\left(e^{\mu} + \frac{\mu-2}{\mu+2}e^{-\mu}\right) \Big| \ (\mu+2)$$

$$\Rightarrow (2-\mu^2)(e^{\mu} - e^{-\mu}) = \mu(e^{\mu} + e^{-\mu})$$

$$\Rightarrow \underbrace{\frac{e^{\mu} - e^{-\mu}}{e^{\mu} + e^{-\mu}}}_{\tanh(\mu)} = \frac{\mu}{2-\mu^2} \quad (*)$$

Conclusion: $(\mu, X(x))$ with $a \neq 0$ eigenpair iff $\mu > 0$ solves $(*)$.

C) Show that eigenval. problem has exactly one negative eigenvalue $\lambda = -\mu^2$.

Sol: From b) this is equivalent to showing that

$$f(\mu) = \tanh(\mu) - \frac{\mu}{2-\mu^2}$$

has exactly one zero for $\mu > 0$.

As $f(\mu) > 0 \quad \forall \ \mu > \sqrt{2}$, zeros must belong to $I = (0, \sqrt{2})$, and zeros of f are zeros of

$$g(\mu) = (2-\mu^2)\tanh(\mu) - \mu \quad \text{on } I.$$

Note

$$g'(\mu) = (2-\mu^2)\tanh'(\mu) - 2\mu\tanh(\mu) - 1$$
$$= (2-\mu^2)(1-\tanh^2(\mu)) - 2\mu\tanh(\mu) - 1$$
$$= 1 - \mu^2 + (\mu^2-2)\tanh^2(\mu) - 2\mu\tanh(\mu)$$

$$g''(\mu) = 2\mu(\tanh^2(\mu) - 1 - \tanh'(\mu))$$
$$+ (\mu^2-2)\cdot 2\tanh(\mu)(1-\tanh^2(\mu))$$
$$- 2\tanh(\mu) < 0 \quad \text{on } (0, \sqrt{2})$$

As $g(0) = 0$, $g'(0) > 0$, $g''|_{(0,\sqrt{2})} < 0$ & $g(\sqrt{2}) < 0$, g has exactly 1 zero

on $(0, \sqrt{2})$, and thus f one zero on $(0,\infty)$.

d) Guess $\mu_0 = 1$.

$$\mu_{n+1} = \mu_n - \frac{f(\mu_n)}{f'(\mu_n)}$$

$$= \mu_n - \frac{\tanh(\mu_n) - \frac{\mu_n}{2-\mu_n^2}}{1 - \tanh^2(\mu_n) - \frac{1}{2-\mu_n^2} - \frac{2\mu_n}{(2-\mu_n^2)^2}}$$

Gives $\mu \approx 0.8712$ & $\lambda = -\mu^2 \approx -0.759$

8.11 Consider

$$u_t = u_{xx} \qquad x \in (0,1),\ t > 0$$

$$u_x(0,t) = 2u(0,t), \quad u_x(1,t) = u(1,t) \qquad t \geq 0$$

Explain why smooth solutions does not in general satisfy

$$\int_0^1 u^2(x,t)\,dx \leq \int_0^1 u^2(x,0)\,dx \qquad t \geq 0.$$

Sol: By Ansatz $u(x,t) = X(x)T(t)$

use obtain eigenval problem

$$-X''(x) = \lambda X(x) \qquad x \in (0,1)$$
$$X'(0) = 2X(0), \quad X'(1) = X(1)$$

& $T'(t) = -\lambda T(t) \qquad t \geq 0$

From 8.10 $\exists\ \lambda = -\mu^2 < 0$ s.t.

$X(x) = e^{\lambda x} + \frac{\mu - 2}{\mu + 2} e^{-\lambda x}$ solves eival PrOb

&

$$T(t) = e^{\mu^2 t}$$

Particular solution: $u(x,t) = e^{\mu^2 t} X(x)$.

$$\Rightarrow \int_0^1 u^2(x,t)\,dx = e^{2\mu^2 t} \int_0^1 X^2(x)\,dx$$
$$> \int_0^1 X^2(x)\,dx = \int_0^1 u^2(x,0)\,dx.$$

Ex 8.14 a) Consider

$$u_t = (pu_x)_x - q u_x \qquad x \in (0,1),\ t \geq 0$$
$$u(0,t) = u(1,t) = 0 \qquad t \geq 0$$
$$u(x,0) = f(x) \qquad x \in (0,1)$$

where $p(x) \geq \alpha > 0$ & $q(x) \geq 0$ $\forall\ x \in [0,1]$

& smooth. Prove for smooth solutions that

$E(t) := \int_0^1 u^2(x,t)\,dx \leq \int_0^1 f^2(x)\,dx \qquad \forall\ t \geq 0.$