use obtain eigenval problem

$$-X''(x) = \lambda X(x) \qquad x \in (0,1)$$
$$X'(0) = 2X(0), \quad X'(1) = X(1)$$

& $T'(t) = -\lambda T(t) \qquad t \geq 0$

From 8.10 $\exists\ \lambda = -\mu^2 < 0$ s.t.

$X(x) = e^{\lambda x} + \frac{\mu - 2}{\mu + 2} e^{-\lambda x}$ solves eival PrOb

&

$$T(t) = e^{\mu^2 t}$$

Particular solution: $u(x,t) = e^{\mu^2 t} X(x)$.

$$\Rightarrow \int_0^1 u^2(x,t)\,dx = e^{2\mu^2 t} \int_0^1 X^2(x)\,dx$$
$$> \int_0^1 X^2(x)\,dx = \int_0^1 u^2(x,0)\,dx.$$

Ex 8.14 a) Consider

$$u_t = (pu_x)_x - q u_x \qquad x \in (0,1),\ t \geq 0$$
$$u(0,t) = u(1,t) = 0 \qquad t \geq 0$$
$$u(x,0) = f(x) \qquad x \in (0,1)$$

where $p(x) \geq \alpha > 0$ & $q(x) \geq 0$ $\forall\ x \in [0,1]$

& smooth. Prove for smooth solutions that

$E(t) := \int_0^1 u^2(x,t)\,dx \leq \int_0^1 f^2(x)\,dx \qquad \forall\ t \geq 0.$

Sol:

$$E'(t) = 2\int_0^1 u u_t \, dx$$

$$= 2\int_0^1 u((pu_x)_x - qu)\, dx$$

$$= \underbrace{2[puu_x]_0^1}_{=0} - 2\int_0^1 \underbrace{p(x)}_{\geq \alpha > 0}(u_x)^2 + \underbrace{q}_{\geq 0} u^2 \, dx$$

$$\leq 0.$$

$$\Rightarrow E(t) \leq E(0).$$

8.17 Consider sequence $\{f_N\}_{N=1}^{\infty} \subset C[-1,1]$

$$f_N(x) = N^{3/2} x e^{-(Nx)^2}.$$

a) Show that $f_N(x) \to 0 \quad \forall x \in [-1,1]$ (pointwise)

Sol: For $x=0$, $f_N(x) = 0 \ \forall N$ so $f_N(0) \to 0$.

For any $x \neq 0$,

$$\lim f_N(x) = \lim_{N\to\infty} \frac{N^{3/2} x}{e^{(Nx)^2}} \overset{\text{l'Hôpital}}{=} \lim_{N\to\infty} \frac{\frac{3}{2} N^{1/2} x}{2Nx^2 e^{(Nx)^2}} = 0.$$

b) Show that f_N does not converge to 0 in MSS.

Sol: $\|f_N - 0\|^2 = \int_{-1}^{1} N^3 x^2 e^{-2(Nx)^2} \, dx \quad \Big| \ Y = Nx$

$$= \int_{-N}^{N} Y^2 e^{-2Y^2} \, dY \to \int_{-\infty}^{\infty} Y^2 e^{-2Y^2} \, dY > 0 \text{ as } N \to \infty.$$

(Problem $f_N(\frac{1}{N}) = \sqrt{N}\, e^{-1} \to \infty$, so $\nexists\, g \in \mathcal{P}$ s.t. $|f_N| \le g$ for all N, meaning dominate conv thm does not apply.)

8.19 a) Show that

$$\int_{-1}^{1} \sin(k\pi x)\, dx = 0 \quad \text{for} \quad k = 1, 2, \ldots$$

Q: Why does this imply that $\{\sin(k\pi x)\}_{k=1}^{\infty}$ is not complete on $\mathcal{P}[-1,1]$?

Sol: $\int_{-1}^{1} \sin(k\pi x)\, dx = \left[\frac{-\cos(k\pi x)}{k\pi}\right]_{-1}^{1} = \frac{(-1)^k - (-1)^k}{k\pi} = 0$

If $\{\sin(k\pi x)\}_{k=1}^{\infty}$ were complete, then for $f \equiv 1 \in \mathcal{P}$ one would have by Parseval's id

$$\sum_{k=1}^{\infty} b_k^2 = \|1\| = \int_{-1}^{1} 1^2\, dx = 2.$$

But $b_k = \int_{-1}^{1} \sin(k\pi x)\, dx = 0 \Rightarrow \sum_{k=1}^{\infty} b_k^2 = 0.$

Show $\{\cos(k\pi x)\}_{k=0}^{\infty}$ not complete on $\mathcal{P}[-1,1]$.

b) Take e.g. $f(x) = x$. Then

$$a_k = \int_{-1}^{1} x \cos(k\pi x)\, dx = 0 \quad \forall\, k$$

$\{\cos(k\pi x)\}_{k=0}^{\infty}$ complete would imply

$$\underbrace{\frac{a_0^2}{2} + \sum_{k=1} a_k^2}_{=0} = \|f\|^2 = 1,$$

so not complete.

Ex 8.20 Let $\{X_k\}_{k=1}^{\infty} \subset \mathcal{P}[a,b] \setminus \{0\}$ be orthogonal. Let $f \in \mathcal{P}[a,b]$ and

$$S_N(f)(x) = \sum_{k=1}^{N} c_k X_k(x), \qquad c_k = \frac{\langle f, X_k \rangle}{\langle X_k, X_k \rangle}$$

Let $P_N(x) = \sum_{k=1}^{N} \alpha_k X_k$ for some $\alpha_k \in \mathbb{R}$.

Show that $\|f - S_N(f)\| \leq \|f - P_N\|$

Sol: Shorthand $S_N = S_N(f)$. And recall $\langle f - S_N, Q_N \rangle = 0$ for any $Q_N \in \operatorname{span}\{X_1, \dots, X_N\}$.

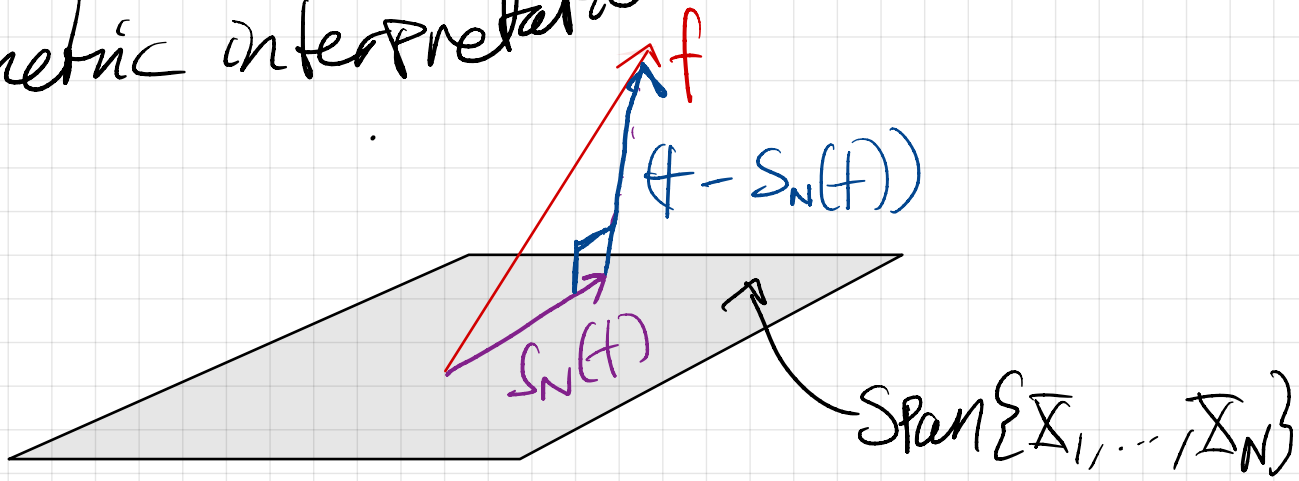
$$\|f - P_N\|^2 = \langle f - P_N, f - P_N \rangle$$

$$= \langle f - S_N + (S_N - P_N), f - S_N + (S_N - P_N) \rangle$$

$$= \|f - S_N\|^2 + 2\underbrace{\langle f - S_N, S_N - P_N \rangle}_{=0} + \|S_N - P_N\|^2$$

$$\geq \|f - S_N\|^2.$$

Geometric interpretation



Ex 9.2 Let $f_N(x) = e^{-x/N}$ $x \in [-1,1]$

Show that $\lim_{N\to\infty} \|f_N - 1\|_\infty = 0$ on $[-1,1]$.

Sol: By Mean-value thm,

$$e^{-y} = e^0 - e^\theta y, \text{ for some } \theta \in (-y, y)$$

Hence for all $y \in [-1,1]$,

$$|e^{-y} - 1| \le e^1 |y|$$

$\Rightarrow$ So for any $x \in [-1,1]$ & $N \ge 1$,

$$|e^{-x/N} - 1| \le e^1 \left|\frac{x}{N}\right| \le \frac{e^1}{N}$$

$$\Rightarrow \sup_{x\in[-1,1]} |e^{-\frac{x}{N}} - 1| \le \frac{e^1}{N} \to 0 \text{ as } N \to \infty.$$

Ex 9.5 Let $f_N(x) = \frac{x}{1+Nx^2}$ $x \in [-1,1]$

a) Find pointwise limits

$$f(x) = \lim_{N\to\infty} f_N(x) \quad \& \quad g(x) = \lim_{N\to\infty} f_N'(x)$$

Sol: $x = 0 \Rightarrow f_N(0) = 0 \ \forall N \ge 1 \Rightarrow f(0) = 0$

$$x \neq 0 \Rightarrow \lim_{N\to\infty} \frac{x}{1+Nx^2} = \frac{x}{\infty} = 0$$

$$\Rightarrow f(x) = 0 \quad \forall x \in [-1,1].$$

$$f_N'(x) = \frac{1+Nx^2-2Nx^2}{(1+Nx^2)^2} = \frac{1-Nx^2}{(1+Nx^2)^2}$$

$$f_N'(0) = 1 \quad \forall N \geq 0 \Rightarrow g'(0) = 1$$

$x \neq 0 \Rightarrow$

$$f_N'(x) = \frac{1-Nx^2}{(1+Nx^2)^2} \to 0 \quad \text{as } N \to \infty$$

$$\Rightarrow g(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$$

b) Show that $f' \neq g$

Sol: $f \equiv 0 \Rightarrow f' \equiv 0$ and $1 = g(0) \neq f'(0)$.

c) Why does this not contradict Prop 9.3?

Sol: Take $[a,b] = [-1,1]$. Prop 9.3 says if $\{f_N\} \subset C^1[-1,1]$ converges uniformly to $f \in C[-1,1]$ and f_N' uniformly to $g \in C[-1,1]$, then $f' = g$ on $[-1,1]$. But f_N' does not converge uniformly to g, as

$$\sup_{x\in[-1,1]} |f_N'(x) - g(x)| \le |f_N'(\tfrac{1}{N})| = \frac{1-\frac{1}{N}}{(1+\frac{1}{N})^2}$$

$$\to 1 \text{ as } N\to\infty.$$

9.10

a) Use full Fourier series of $f(x)=x^2$ & Parseval's id to show that

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90},$$

Recall: $$S_\infty(f) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}\cos(k\pi x)$$

Sol: $a_0 = \frac{2}{3}$, $a_k = \frac{(-1)^k\cdot 4}{\pi^2 k^2}$ $\Big\}$ $k\ge 1$.

$b_k = 0$

Parseval:

$$\underbrace{\frac{a_0^2}{2} + \sum_{k=1}^{\infty} a_k^2}_{\text{I}} = \underbrace{\int_{-1}^{1} x^4\,dx}_{\text{II}}$$

$$\frac{2}{9} + \sum_{k=1}^{\infty} \frac{16}{\pi^4}\cdot\frac{1}{k^4} = \frac{2}{5}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{16} \cdot \frac{8}{45} = \frac{\pi^4}{90}$$

Compute $\sum_{k=1}^{\infty} \left(\frac{1}{2k-1}\right)^4$ by Parseval & $f(x)=|x|$.

b) Sol: As $f(x)=|x|$ is even,

$$a_0 = \int_{-1}^{1} |x|\,dx = 1, \quad k \geq 1:$$

$$a_k = \int_{-1}^{1} |x| \cos(k\pi x)\,dx = 2\int_0^1 x\cos(k\pi x)\,dx$$

$$= \frac{2}{k\pi}\left[x\sin(k\pi x)\right]_0^1 - \frac{2}{k\pi}\int_0^1 \sin(k\pi x)\,dx$$

$$= \frac{2}{(k\pi)^2}\left[\cos(k\pi x)\right]_0^1 = \frac{2}{(k\pi)^2}\underbrace{\left((-1)^k - 1\right)}_{= \begin{cases} -2 & k \text{ odd} \\ 0 & k \text{ even} \end{cases}}$$

&

$$b_k = \int_{-1}^{1} \underbrace{|x|}_{\text{even}} \underbrace{\sin(k\pi x)}_{\text{odd}}\,dx = 0$$

odd

odd

$$S_N(f) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}\cos\left((2k-1)\pi x\right)$$

$$\Rightarrow \frac{1}{2} + \sum_{k=1}^{\infty} a_k^2 = \int_{-1}^{1} x^2 \, dx = \frac{2}{3}$$

$$\Rightarrow \frac{16}{\pi^4} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^4} = \frac{1}{6}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{(2k-1)^4} = \frac{\pi^4}{96} \quad \blacksquare$$

Ex 9.12 $f(x) = x^3 - x$ on $[-1,1]$

a) Show that $f \in C_p^1[-1,1]$ but $f \notin C_p^2[-1,1]$.

sol: $f(-1) = f(1) = 0$ & $f'(x) = 3x^2 - 1$

$\Rightarrow f'(-1) = f'(1) = 2, \Rightarrow f \in C_p^1[-1,1]$

& $f''(x) = 6x$

$\Rightarrow f''(-1) = -6 \neq f''(1) = 6. \Rightarrow f \notin C_p^2.$

b) Explain why $\sum_{k=1}^{\infty} k^4(a_k^2+b_k^2) < \infty$

Sol Since $f \in C_p^1[-1,1]$ & $f^{(2)} \in \mathcal{R}[-1,1]$, it holds by Thm 9.5 that

$$\sum_{k=1}^{\infty} k^4(a_k^2+b_k^2) = \pi^{-4}\|f^{(2)}\|^2 < \infty$$

If it were also true that $\sum_{k=1}^{\infty} k^6(a_k^2+b_k^2) < \infty$, then Thm 9.6 would imply that

$$\exists\, g \in C_p^2[-1,1] \text{ with}$$

fourier series

$$S_\infty(g) = \hat{a}_0 + \sum_{k=1}^{\infty} a_k \cos(k\pi x) + b_k \sin(k\pi x)$$

This means $g - f = \frac{\hat{a}_0 - a_0}{2} = c$

functions only differ by a constant which would imply the contradiction that also $f = g - c$ belongs to $C_p^2[-1,1]$.

c) $f''(x) = 6x$ ← notation

$$S_\infty(f'') = \sum_{k=1}^{\infty} b_k'' \sin(k\pi x)$$

with $b_k'' = 6\int_{-1}^{1} \sin(k\pi x)\, x\, dx$

$$= 12\int_0^1 \sin(k\pi x)\, x\, dx$$

$$= -\frac{12}{k\pi}\left[\cos(k\pi x)\, x\right]_0^1 + \frac{12}{k\pi}\int_0^1 \cos(k\pi x)\, dx$$

$$= \frac{12}{k\pi}(-1)^{k+1}$$

$f' \in C_p^0[-1,1]$ & $f'' \in P[-1,1] \Rightarrow$

$$S_\infty(f') = \frac{a_0'}{2} + \sum_{k=1}^{\infty} a_k' \cos(k\pi x)$$

with $-k\pi a_k' = b_k'' \Rightarrow a_k' = \frac{12\cdot(-1)^k}{(k\pi)^2}$

$a_0' = \int_{-1}^{1}(3x^2-1)\, dx$

& $S_\infty(f) = \sum_{k=1}^{\infty} b_k \sin(k\pi x)$

with $k\pi b_k = a_k' \Rightarrow \frac{12\cdot(-1)^k}{(k\pi)^3}$

Conclusion:

$$\sum_{k=1}^{\infty} k^6 b_k^2 = \frac{12^2}{\pi^6} \sum_{k=1}^{\infty} 1 = \infty$$

$$\& \sum_{k=1}^{\infty} k^4 b_k^2 = \frac{12^2}{\pi^6} \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty,$$

agrees with parts a & b)

Ex 9.17 (Relates to proof of Thm 9.4)

a) Let $H(x)$ be 2-periodic extension of

$$H(x) = \begin{cases} 1 & x \in (0,1] \\ 0 & x \in (-1,0] \end{cases}$$

and for $\delta \in (0, \frac{1}{2})$ $x \in [-1,1]$, let

$$H_\delta(x) = \frac{1}{2\delta} \int_{-\delta}^{\delta} H(x+y)\, dy$$

Show that $\int_{-1}^{1} (H_\delta - H)^2 dx \to 0$ as $\delta \downarrow 0$.

Sol: Write $H(x) = \frac{1}{2\delta} \int_{-\delta}^{\delta} H(x)\, dy$.

Then $\int_{-1}^{1} (H_\delta - H)^2(x)\,dx$

$$= \frac{1}{4\delta^2} \int_{-1}^{1} \left(\int_{-\delta}^{\delta} 1 \cdot (H(x+y) - H(x))\,dy \right)^2 dx$$

$$\leq \frac{1}{4\delta^2} \int_{-1}^{1} \underbrace{\int_{-\delta}^{\delta} 1^2\,dy}_{2\delta} \int_{-\delta}^{\delta} (H(x+y) - H(x))^2\,dy\,dx$$

$(H(x+y) - H(x))^2 \leq$ when $y \in (-\delta, \delta)$

$$\begin{cases} 1 & |x| > 1-\delta \text{ or } |x| < \delta \\ 0 & \text{else} \end{cases}$$

$$\leq \frac{1}{2\delta} \int_{[1-\delta,1] \cup [-\delta,\delta] \cup [-1,-1+\delta]} \int_{-\delta}^{\delta} 1\,dy\,dx$$

$$\leq \int_{[1-\delta,1] \cup [-\delta,\delta] \cup [-1,-1+\delta]} 1\,dx$$

$= 4\delta \to 0$ as $\delta \to 0$

b) Let $g \in C_p^0[-1,1]$, and show that

$$\lim_{\delta \downarrow 0} \|g - g_\delta\|^2 = 0.$$

Sol: As in a) we have that

$$g(x) - g_\delta(x) = \frac{1}{2\delta} \int_{-\delta}^{\delta} g(x) - g(x+y)\, dy.$$

Let $C_\delta = \max_{(x,y) \in [-1,1] \times [-\delta,\delta]} |g(x) - g(x+y)|$

Since g is uniformly cont. on $[-1,1]$, $C_\delta \to 0$ as $\delta \downarrow 0$, and

$$\int_{-1}^{1} (g(x) - g_\delta(x))^2 dx$$

$$= \frac{1}{4\delta^2} \int_{-1}^{1} \left(\int_{-\delta}^{\delta} g(x) - g(x+y)\, dy \right)^2 dx$$

$$\leq \frac{1}{4\delta^2} \int_{-1}^{1} \int_{-\delta}^{\delta} 1\, dy \int_{-\delta}^{\delta} (g(x) - g(x+y))^2 dy\, dx$$

$$\leq \frac{1}{2\delta} \int_{-1}^{1} \int_{-\delta}^{\delta} C_\delta^2\, dy\, dx = 2 C_\delta^2 \to 0$$

as $\delta \downarrow 0$.

c) Explain why any 2-periodic $f \in \mathcal{P}[-1,1]$ can be written

$$f(x) = g(x) + \sum_{j=1}^{M} c_j H(x - x_j)$$

Sol: f has M discontinuity points $x_1, \dots, x_M$. Let $c_j := f(x_j+) - f(x_j-)$

$$g(x) := f(x) - \sum_{j=1}^{M} c_j H(x - x_j)$$

Then g is 2-periodic, since f & $H(\cdot - x_j)$ all are 2-periodic. Moreover $g \in C[-1,1]$ since it is only possible that $g(x+) \neq g(x-)$ at $x \in \{x_1, \dots, x_M\}$ but

$$g(x_i+) = f(x_i+) - \sum_{j=i}^{M} c_j$$

& $$g(x_i-) = f(x_i-) - \sum_{j=i+1}^{M} c_j$$

$$\Rightarrow g(x+) - g(x_i-) = f(x_i+) - f(x_i-) - c_i$$

$$= 0 \quad \text{for } i = 1, \dots, M.$$

d) Show that

$\|f - f_\delta\| \to 0$ as $\delta \downarrow 0$ for any $f \in \mathcal{P}[-1,1]$.

Sol: By c), $\exists\, g \in C_p^0[-1,1]$ s.t.

$$f(x) = g(x) + \sum_{i=1}^{M} c_j H(x - x_j)$$

By linearity

$$f_\delta(x) = \frac{1}{2\delta} \int_{-\delta}^{\delta} g(x+y) + \sum_{i=1}^{M} c_j H(x - x_j + y)\, dy$$

$$= g_\delta(x) + \sum_{i=1}^{M} c_j H_\delta(x - x_j)$$

$$\Rightarrow \|f - f_\delta\| \le \|g - g_\delta\| + \sum_{i=1}^{M} |c_j| \, \|H(\cdot - x_j) - H_\delta(\cdot - x_j)\|$$

$\to 0$ as $\delta \downarrow 0$ by b) & a).

10.1 Let $f \in C[0,\ell]$ and $f' \in \mathcal{P}[0,\ell]$ with $f(0) = f(\ell) = 0$. Show that

$$\int_0^\ell f^2(x)\,dx \le \frac{\ell^2}{\pi^2} \int_0^\ell (f'(x))^2\,dx$$

Sol: Let $\tilde{f}(x) = \begin{cases} f(x) & x \in [0,\ell] \\ -f(-x) & x \in [-\ell, 0) \end{cases}$

and $g(x) = \tilde{f}(\ell x) \qquad x \in [-1,1]$

Then $g(x) := \tilde{f}(\ell x) = -\tilde{f}(-\ell x)$

$= -g(-x)$

is odd, $g \in C_p^0[-1,1]$ and $g' \in \mathcal{P}[-1,1]$.

Hence $g(x) = \sum_{k=1}^{\infty} b_k \sin(k\pi x)$

for some $\{b_k\}_{k=1}^{\infty}$ and by Thm 9.5

$$g'(x) = \sum_{k=1}^{\infty} k\pi b_k \cos(k\pi x)$$

So $\int_{-1}^{1} (g'(x))^2 \, dx \overset{\text{Parseval}}{=} \sum_{k=1}^{\infty} (k\pi)^2 b_k^2$

$$\geq \pi^2 \sum_{k=1}^{\infty} b_k^2 \overset{\text{Parseval}}{=} \pi^2 \int_{-1}^{1} (g(x))^2 \, dx$$

Combined with

$$\int_{-1}^{1} (g'(x))^2 dx = 2\int_{0}^{1} \ell^2 (f'(\ell x))^2 dx$$

$$= 2\ell \int_0^{\ell} (f'(y))^2 dy \qquad \boxed{y = \ell x}$$

and

$$\int_{-1}^{1} (g(x))^2 dx = 2\int_0^1 (f(\ell x))^2 dx$$

$$= \frac{2}{\ell} \int_0^{\ell} (f(y))^2 dy$$

we get that

$$\int_0^{\ell} (f(y))^2 dy \leq \frac{\ell^2}{\pi^2} \int_0^{\ell} (f'(y))^2 dy$$

10.2 Let $f \in C_p^0[-1,1]$ & $f' \in P[-1,1]$, and assume that $\int_{-1}^{1} f(x)\,dx = 0$ $(*)$

a) Show that $f' \equiv 0 \Rightarrow f \equiv 0$.

Sol: $f' \equiv 0 \Rightarrow f \equiv c$

$(*) \Rightarrow \int_{-1}^{1} c\,dx = 0 \Rightarrow c = 0$.

b) Establish Poincaré's ineq

$$\int_{-1}^{1} f^2(x)\,dx \le \frac{1}{\pi^2} \int_{-1}^{1} (f'(x))^2\,dx$$

Sol By properties of f $\exists$ $\{a_k\}, \{b_k\}$ s.t.

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\pi x) + b_k \sin(k\pi x)$$

and $a_0 = \int_{-1}^{1} f(x)\,dx \overset{(*)}{=} 0$.

$$f'(x) = \sum_{k=1}^{\infty} (k\pi)\left(b_k \cos(k\pi x) - a_k \sin(k\pi x)\right)$$

$$\int_{-1}^{1} (f'(x))^2\,dx \overset{\text{Parseval}}{=} \sum_{k=} (k\pi)^2 (b_k^2 + a_k^2)$$

$$\geq \pi^2 \sum_{k=1}^{\infty} a_k^2 + b_k^2 \overset{\text{Parseval}}{=} \int_{-1}^{1} f^2(x)\,dx$$

10.3 Find an $f \in C^1[0,1]$ with $f(0)=0$ s.t.

$\|f\| > \frac{1}{\pi}\|f'\|$. Why does it not contradict Corollary 10.1:

$\|g\| \leq \frac{1}{\pi}\|g'\|$ $\forall$ $g \in C^0_{p,0}[0,1]$ with $g' \in \mathcal{P}[0,1]$?

Sol: Take $f(x) = x$. Then

$$\|f\| = \sqrt{\int_0^1 x^2\,dx} = \sqrt{\frac{1}{3}}, \text{ while}$$

$$\|f'\| = \int_0^1 1^2\,dx = 1.$$

$$\Rightarrow \frac{\|f'\|}{\pi} < \frac{1}{\sqrt{3}} = \|f\|.$$

Does not contradict Cor 10.1 as

$f(1) = 1 \Rightarrow f \notin C^0_{p,0}[0,1]$.

10.7 Consider

$$u_t = u_{xx} \qquad x \in (-1,1), \quad t > 0$$

$$u(-1,t) = u(1,t), \quad u_x(-1,t) = u_x(1,t) \qquad t \geq 0$$

$$u(x,0) = f(x) \qquad x \in (-1,1)$$

Let $f \in \mathcal{P}[-1,1]$,

$$S_N(f) = \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos(k\pi x) + b_k \sin(k\pi x)$$

$$u^N(x,t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} e^{-(k\pi)^2 t}\left(a_k \cos(k\pi x) + b_k \sin(k\pi x)\right)$$

with formal sol $u = u^\infty$.

a) Show that $u(\cdot,t) \in C_p^\infty[-1,1]$ for any $t > 0$.

Sol: For $t > 0$, $\{u^N(\cdot,t)\}_{N \geq 1}$ is a series of functions and for any $m \in \mathbb{N}$

$$\sum_{k=1}^{\infty} \underbrace{k^{2m} e^{-2(k\pi)^2 t}}_{\text{Lem 10.2}} \left(a_k^2 + b_k^2\right)$$

$$\leq M_{m,t} = \left(\frac{m}{2t\pi^2}\right)^m e^{-m}$$

$$\leq M_{m,t} \sum_{k=1}^{\infty} a_k^2 + b_k^2 \leq M_{m,t} \|f\|^2 < \infty.$$

Thm 9.6 $\Rightarrow u^N(\cdot,t) \to u(\cdot,t)$ uniformly on $[-1,1]$ as $N \to \infty$ and that $u(\cdot,t) \in C_p^{m-1}[-1,1]$ for any $m \in \mathbb{N}$.

Hence $u(\cdot,t) \in C_p^{\infty}[-1,1]$.

b) Let $E(t) = \int_{-1}^{1} u^2(x,t)\,dx$

Explain why in general it is not true that $\lim_{t\to\infty} E(t) = 0$.

Sol: Take $f \equiv 1$. Then $u \equiv 1$ is the unique solution, and $E \equiv 2$. $\forall\, t \geq 0$.

c) Show that if $\int_{-1}^{1} f(x)\,dx = 0$, then $E(t) \leq e^{-2\pi^2 t} E(0)$

Sol: Then $a_0 = \int_{-1}^{1} f(x)\,dx = 0$ so

$$E(t) = \|u(\cdot,t)\|^2 \qquad \text{(Parseval)}$$
$$= \sum_{k=1}^{\infty} e^{-2(k\pi)^2 t}\left(a_k^2 + b_k^2\right)$$

$$\leq e^{-2\pi^2 t} \sum_{k=1}^{\infty} a_k^2 + b_k^2$$

$$\leq e^{-2\pi^2 t} \|f\|^2 = e^{-2\pi^2 t} E(0).$$

Ex 10.8 Let $f \in \mathcal{B}[0,1]$, with $S_N(f) = \sum_{k=1}^{N} c_k \sin(k\pi x)$. For $t > 0$, set $f_t^N(x) = \sum_{k=1}^{N} c_k e^{-k t} \sin(k\pi x)$

a) Show that $f_t \in C^{\infty}_{P,0}[0,1]$ for any $t > 0$.

Sol: Let $t > 0$ and let $\tilde{f}_t^N$ be odd extension of f_t.

Then for any $m \in \mathbb{N}$

$$\sum_{k=1}^{\infty} \underbrace{k^{2m} e^{-2kt}}_{= M_{m,t}} c_k^2 \leq M_{m,t} \sum_{k=1}^{\infty} c_k^2$$

$$\overset{\text{Parseval}}{=} 2 M_{m,t} \int_0^1 f(x)\, dx < \infty$$

Where by Lemma 10.2

$$M_{m,t} = \left(\frac{m}{t}\right)^{2m} e^{-2m}$$

Thm 9.6 implies that

$\tilde{f}_t^N \to \tilde{f}_t^\infty =: \tilde{f}_t$ uniformly on $[-1,1]$

& that $\tilde{f}_t \in C_p^m[-1,1]$ for any $m \in \mathbb{N}$

Moreover $\tilde{f}_t(-x) = \lim_{N\to\infty} \underbrace{\tilde{f}_t^N(-x)}_{\text{odd}}$

$$= -\lim \tilde{f}_t^N(x)$$

$$= -\tilde{f}_t(x)$$

so $\tilde{f}$ is odd.

$\tilde{f}_t$ odd & $\tilde{f}_t \in C_p^m[-1,1]$ $\forall m \geq 1$ $\Rightarrow$ Lemma 10.1

$f_t := \tilde{f}_t|_{[0,1]}$ satisfies $f_t \in C_{p,0}^m[0,1]$

$\forall m \geq 1$.

(b) Show that $\lim_{t \downarrow 0} \|f_t - f\| = 0$

Sol:

And

$$\|f - f_t\| \le \|f - f^N\| + \|f^N - f_t^N\| + \|f_t^N - f_t\| =: I_N + II_{N,t} + III_{N,t}$$

Where

$$I_N^2 \overset{\text{Parseval}}{=} \sum_{k=N+1}^{\infty} c_k^2 \to 0 \quad \text{as } N \to \infty$$

$$II_{N,t}^2 = \sum_{k=1}^{N} c_k^2 (1 - e^{-kt})^2$$

$$\le (1 - e^{-Nt})^2 \, 2\|f\|^2 \to 0 \quad \text{as } t \downarrow 0$$

$$III_{N,t}^2 = \sum_{k=N+1}^{\infty} c_k^2 e^{-2kt} \le I_N^2 \to 0 \text{ as } N \to \infty.$$

Consequently,

$$\lim_{t \downarrow 0} \|f - f_t\| = \lim_{N \to \infty} \lim_{t \downarrow 0} \|f - f_t\| = 0.$$