

Solutions to the exam in MAT3400/4400, Spring 2026

Problem 1

Find the limits of the following numerical sequences $(c_n)_{n=1}^{\infty}$ as $n \rightarrow +\infty$. Remember to justify your answers.

(a)

$$c_n = \int_0^{+\infty} \frac{1}{1+x^2} \sin\left(\frac{x+\pi n}{2n}\right) dx.$$

(b)

$$c_n = \int_n^{+\infty} e^{-\sqrt{x}} \cos(x^n) dx.$$

Solution:

(a) Consider the functions $f_n(x) = \frac{1}{1+x^2} \sin\left(\frac{x+\pi n}{2n}\right)$. Since $\frac{x+\pi n}{2n} \rightarrow \frac{\pi}{2}$ as $n \rightarrow +\infty$, we have $f_n(x) \rightarrow \frac{1}{1+x^2}$. We also have $|f_n(x)| \leq \frac{1}{1+x^2}$. Since the last function is integrable on $[0, +\infty)$, by the dominated convergence theorem we conclude that

$$c_n = \int_0^{\infty} f_n(x) dx \rightarrow \int_0^{\infty} \frac{dx}{1+x^2} = \arctan x \Big|_0^{\infty} = \frac{\pi}{2}.$$

(b) Consider the functions $f_n(x) = \mathbf{1}_{[n, +\infty)}(x) e^{-\sqrt{x}} \cos(x^n)$. Then $f_n(x) \rightarrow 0$ as $n \rightarrow +\infty$ and $|f_n(x)| \leq e^{-\sqrt{x}}$. Since the last function is integrable on $[0, +\infty)$, we conclude that

$$c_n = \int_0^{\infty} f_n(x) dx \rightarrow 0.$$

Problem 2

Assume $(X, \mathcal{B}, \mu)$ is a σ -finite measure space.

(a) Assume ν is another σ -finite measure on $(X, \mathcal{B})$ such that $\nu \ll \mu$. Explain why for any measurable function $f: X \rightarrow [0, +\infty]$ we have

$$\int_X f d\nu = \int_X f \frac{d\nu}{d\mu} d\mu.$$

(b) Show that if ν is as above and η is another σ -finite measure on $(X, \mathcal{B})$ such that $\eta \ll \nu$, then

$$\frac{d\eta}{d\mu} = \frac{d\eta}{d\nu} \cdot \frac{d\nu}{d\mu} \quad \mu\text{-a.e.}$$

Solution:

(a) The equality holds for all $f = \mathbf{1}_A$ ($A \in \mathcal{B}$) by the definition of the Radon–Nikodym derivative. Then it holds for all positive simple measurable functions f by linearity of the integral. Finally, to see that it holds for all positive measurable functions f we take a sequence of positive simple measurable functions f_n such that $f_n \nearrow f$ pointwise and apply the monotone convergence theorem.

(b) Take any $A \in \mathcal{B}$. Applying the definition of the Radon–Nikodym derivative and then (a) we get

$$\int_X \mathbf{1}_A d\eta = \int_X \mathbf{1}_A \frac{d\eta}{d\nu} d\nu = \int_X \mathbf{1}_A \frac{d\eta}{d\nu} \frac{d\nu}{d\mu} d\mu.$$

This gives the required conclusion by essential uniqueness of Radon–Nikodym derivatives.

Problem 3

Recall that $\mathcal{S}(\mathbb{R})$ denotes the space of Schwartz functions on $\mathbb{R}$.

(a) Determine whether the function $f(x) = e^{-x^2} \sin(e^{x^2})$ lies in $\mathcal{S}(\mathbb{R})$.

(b) Show that there is no function $f \in \mathcal{S}(\mathbb{R})$ such that $f * g = g$ for all $g \in \mathcal{S}(\mathbb{R})$, where $*$ denotes convolution. Hint: Fourier transform.

Solution:

(a) We have

$$f'(x) = -2xe^{-x^2} \sin(e^{x^2}) + 2x \cos(e^{x^2}).$$

We see that the function f' is unbounded, hence f is not a Schwartz function.

(b) Assume there is such a function f . Since the Fourier transform maps the convolution of two integrable functions into the product of their Fourier transforms up to a scalar factor, namely, the factor 2π in the conventions used in the class, we then have $2\pi \hat{f}\hat{g} = \hat{g}$ for all $g \in \mathcal{S}(\mathbb{R})$. Since for every point x we can find some $g \in \mathcal{S}(\mathbb{R})$ such that $\hat{g}(x) \neq 0$ (for example, we can take $g(x) = e^{-x^2}$), it follows that $\hat{f} = \frac{1}{2\pi}$. Since $\hat{f}$ is itself a Schwartz function, while any nonzero constant function is not, we get a contradiction.

Problem 4

Assume $(X, \mathcal{B}, \mu)$ is a measure space.

(a) Formulate Hölder's inequality. Using it prove the following generalization: if $p, q, r \in [1, \infty)$ are such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, then for any measurable functions $f, g: X \rightarrow \mathbb{C}$ we have

$$\left(\int_X |fg|^r d\mu \right)^{1/r} \leq \left(\int_X |f|^p d\mu \right)^{1/p} \left(\int_X |g|^q d\mu \right)^{1/q}.$$

(b) Use induction to prove the following even more general inequality: if $p_1, \dots, p_n, r \in [1, \infty)$ are such that $\frac{1}{p_1} + \dots + \frac{1}{p_n} = \frac{1}{r}$, then for any measurable functions $f_1, \dots, f_n: X \rightarrow \mathbb{C}$ we have

$$\left(\int_X |f_1 \dots f_n|^r d\mu \right)^{1/r} \leq \prod_{k=1}^n \left(\int_X |f_k|^{p_k} d\mu \right)^{1/p_k}.$$

Solution:

(a) Hölder's inequality states that if $f, g: X \rightarrow \mathbb{C}$ are two measurable functions, $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_X |fg| d\mu \leq \left(\int_X |f|^p d\mu \right)^{1/p} \left(\int_X |g|^q d\mu \right)^{1/q}.$$

Applying this inequality to the functions $|f|^r, |g|^r$ and the numbers p/r and q/r instead of p and q , we get

$$\int_X |fg|^r d\mu \leq \left(\int_X |f|^p d\mu \right)^{r/p} \left(\int_X |g|^q d\mu \right)^{r/q}.$$

Raising this to the power $1/r$ we get the required inequality.

(b) The inequality is obvious for $n = 1$. For $n = 2$ it is proved in (a). The general case is proved by induction on n . For the induction step, assume the inequality is true for some $n \geq 2$ and let us

prove it for $n+1$. Let $p \in (1, \infty)$ be such that $\frac{1}{p_1} + \dots + \frac{1}{p_n} = \frac{1}{p}$, so that $\frac{1}{p} + \frac{1}{p_{n+1}} = \frac{1}{r}$. Applying (a) and the inductive assumption we then get

$$\begin{aligned} \left(\int_X |f_1 \dots f_{n+1}|^r d\mu \right)^{1/r} &\leq \left(\int_X |f_1 \dots f_n|^p d\mu \right)^{1/p} \left(\int_X |f_{n+1}|^{p_{n+1}} d\mu \right)^{1/p_{n+1}} \\ &\leq \left(\int_X |f_1|^{p_1} d\mu \right)^{1/p_1} \dots \left(\int_X |f_n|^{p_n} d\mu \right)^{1/p_n} \left(\int_X |f_{n+1}|^{p_{n+1}} d\mu \right)^{1/p_{n+1}}, \end{aligned}$$

which is what we need.

Problem 5

Assume $(X, \mathcal{B}, \mu)$ is a finite measure space.

(a) Show that for all $1 \leq p < q \leq \infty$ we have $L^p(X, \mathcal{B}, d\mu) \supset L^q(X, \mathcal{B}, d\mu)$.

(b) Show that for any $f \in L^\infty(X, \mathcal{B}, \mu)$ we have

$$\|f\|_\infty = \lim_{p \rightarrow \infty} \|f\|_p.$$

Solution:

(a) Assume first that $1 \leq p < q < \infty$. Take $f \in L^q(X, \mathcal{B}, d\mu)$. Let $r \in (1, \infty)$ be such that $\frac{p}{q} + \frac{1}{r} = 1$. Applying Hölder's inequality to $|f|^p$ and $g = \mathbb{1}_X$ we get

$$\int_X |f|^p d\mu \leq \left(\int_X (|f|^p)^{q/p} d\mu \right)^{p/q} \left(\int_X |g|^r d\mu \right)^{1/r} = \left(\int_X |f|^q d\mu \right)^{p/q} \mu(X)^{1/r}.$$

Raising this to the power $1/p$ we obtain

$$\left(\int_X |f|^p d\mu \right)^{1/p} \leq \left(\int_X |f|^q d\mu \right)^{1/q} \mu(X)^{1/p-1/q}.$$

(Alternatively, apply 4(a) to f , $g = \mathbb{1}_X$, $r = p$ and $p = q$.) This shows that $f \in L^p(X, \mathcal{B}, d\mu)$.

Consider now the case $q = \infty$. If $f \in L^\infty(X, \mathcal{B}, \mu)$ and $1 \leq p < \infty$, then

$$\left(\int_X |f|^p d\mu \right)^{1/p} \leq \left(\|f\|_\infty^p \mu(X) \right)^{1/p} = \|f\|_\infty \mu(X)^{1/p}, \quad (1)$$

so $f \in L^p(X, \mathcal{B}, d\mu)$.

(b) Take $f \in L^\infty(X, \mathcal{B}, \mu)$. Then by (1) we have $\|f\|_p \leq \|f\|_\infty \mu(X)^{1/p}$ and hence

$$\limsup_{p \rightarrow \infty} \|f\|_p \leq \|f\|_\infty \lim_{p \rightarrow \infty} \mu(X)^{1/p} = \|f\|_\infty. \quad (2)$$

If $\|f\|_\infty = 0$, there is nothing left to prove, so assume $\|f\|_\infty > 0$. Take $0 < \varepsilon < \|f\|_\infty$ and consider the measurable set

$$A = \{x : |f(x)| > \|f\|_\infty - \varepsilon\}.$$

Then $\mu(A) > 0$ and we have

$$\|f\|_p \geq \left(\int_A |f|^p d\mu \right)^{1/p} \geq (\|f\|_\infty - \varepsilon) \mu(A)^{1/p},$$

hence

$$\liminf_{p \rightarrow \infty} \|f\|_p \geq (\|f\|_\infty - \varepsilon) \lim_{p \rightarrow \infty} \mu(A)^{1/p} = \|f\|_\infty - \varepsilon.$$

Since this is true for all $0 < \varepsilon < \|f\|_\infty$, we thus get

$$\liminf_{p \rightarrow \infty} \|f\|_p \geq \|f\|_\infty,$$

which together with (2) proves that $\lim_{p \rightarrow \infty} \|f\|_p = \|f\|_\infty$.