

Measurable maps

(1)

Proposition

Assume $(X, \mathcal{B})$ is a measurable space, (Y, d) is a metric space. Assume $(f_n: (X, \mathcal{B}) \rightarrow Y)_{n=1}^{\infty}$ is a sequence of measurable maps s.t. $(f_n(x))_{n=1}^{\infty}$ is convergent $\forall x \in X$. Then the map $f: X \rightarrow Y$, $f(x) = \lim_{n \rightarrow \infty} f_n(x)$, is measurable (with respect to $\mathcal{B}$ and $\mathcal{B}(Y)$).

Proof

It suffices to show that if $F \subset Y$ is a closed subset, then $f^{-1}(F) \in \mathcal{B}$. But this follows from the equality

$$(x) \quad f^{-1}(F) = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} f_m^{-1}(\{y: d(y, F) \leq \frac{1}{k}\})$$

where $d(y, F) = \inf \{d(y, z): z \in F\}$.

Note that the function $y \mapsto d(y, F)$ is continuous, since

$$|d(y, F) - d(z, F)| \leq d(y, z),$$

hence $\{y: d(y, F) \leq \frac{1}{k}\}$ is closed.

To prove (x), assume first that $x \in f^{-1}(F)$.

Let n be s.t. $d(f(x), f_n(x)) \leq \frac{1}{k} \quad \forall n \geq n$.

Then $d(f_n(x), F) \leq \frac{1}{k}$, so

$$x \in \bigcap_{n=m}^{\infty} f_n^{-1}(\{y: d(y, F) \leq \frac{1}{k}\}).$$

this proves that $C \subset (x)$.

For the opposite inclusion, take

$$x \in \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} f_m^{-1}(\{y: d(y, F) \leq \frac{1}{k}\}).$$

Fix k . Then $x \in \bigcap_{n=m}^{\infty} f_m^{-1}(\{y: d(y, F) \leq \frac{1}{k}\})$

for some m , that is,

$$d(f_n(x), F) \leq \frac{1}{k} \quad \forall n \geq m.$$

On the other hand, $f_n(x) \rightarrow f(x)$, so there

$$\text{is } m' \text{ s.t. } d(f_n(x), f(x)) \leq \frac{1}{k} \quad \forall n \geq m'.$$

Then, for all $n \geq \max\{m, m'\}$, we have

$$d(f(x), F) \leq d(f(x), f_n(x)) + d(f_n(x), F) \leq \frac{2}{k},$$

so there is $y_n \in F$ s.t.

$$d(f(x), y_n) \leq \frac{3}{k}.$$

Then $f(x) = \lim_{k \rightarrow \infty} y_k \in F$, as F is closed. $\square$

Consider a measurable space $(X, \mathcal{B})$.

Properties of measurable functions:

1) if $f, g: (X, \mathcal{B}) \rightarrow \mathbb{C}$ are measurable,

then $f+g, fg$ ($(fg)(x) = f(x)g(x)$)

and cf ($c \in \mathbb{C}$) are measurable;

2) if $f: (X, \mathcal{B}) \rightarrow \mathbb{C}$ is measurable

and $h: \mathbb{C} \rightarrow \mathbb{C}$ is Borel, then

$h \circ f$ is measurable;

3) if $X = \bigcup_{n=1}^{\infty} A_n$, $A_n \in \mathcal{B}$, and

②

$f|_{A_n} : (A_n, \mathcal{B}_{A_n}) \rightarrow \mathbb{C}$ are measurable for all n , then f is measurable;

4) if $f(x) = \lim_{n \rightarrow \infty} f_n(x) \quad \forall x \in X$ and f_n are measurable for all n , then f is measurable.

Proof

2)-4) follows from similar general statements for measurable maps.

To prove 1), consider the map

$$F: X \rightarrow \mathbb{C}^2, \quad F(x) = (f(x), g(x)).$$

we claim that F is measurable. Indeed, it suffices to show that $F^{-1}(A \times B) \in \mathcal{B}$ $\forall$ Borel sets $A, B \subset \mathbb{C}$, since such $A \times B$ generate $\mathcal{B}(\mathbb{C}^2)$ as a σ -algebra (it suffices to take parallelepipeds). But

$$\begin{aligned} F^{-1}(A \times B) &= \{x : (f(x), g(x)) \in A \times B\} \\ &= f^{-1}(A) \cap g^{-1}(B) \in \mathcal{B}, \end{aligned}$$

so F is measurable.

Then $f+g$ is the composition

$$X \xrightarrow{F} \mathbb{C}^2 \xrightarrow{\quad} \mathbb{C}$$

$$(z_1, z_2) \mapsto z_1 + z_2$$

which is measurable, since the last

map is continuous, hence Borel.
 Similarly f_1 and c_1 are measurable
 as compositions

$$\mathbb{R} \times \mathbb{R} \xrightarrow{F} \mathbb{R}^2 \rightarrow \mathbb{C}$$

$$(a_1, b_1) \mapsto z_1 z_2$$

and $X \xrightarrow{f_1} \mathbb{R} \xrightarrow{c_1} \mathbb{C}$

□

One can similarly prove the following
 property more general than (1) :

If Y_1, Y_2 are separable metric spaces
 (that is, they have dense countable
 subspaces), Z is a metric space,
 $f_i: (X, \mathcal{A}) \rightarrow (Y_i, \mathcal{B}(Y_i))$ ($i=1,2$) are
 measurable maps and $g: Y_1 \times Y_2 \rightarrow Z$
 is Borel, then the map

$$f: X \rightarrow Z, f(x) = g(f_1(x), f_2(x))$$

is measurable.

For the proof we need to show that
 the map $F: X \rightarrow Y_1 \times Y_2, x \mapsto (f_1(x), f_2(x))$
 is measurable. For this it suffices to show
 that $\mathcal{B}(Y_1 \times Y_2)$ is generated by the sets $U_1 \times U_2$,
 with $U_i \subset Y_i$ open, since

$$F^{-1}(U_1 \times U_2) = f_1^{-1}(U_1) \cap f_2^{-1}(U_2)$$

Let $(y'_n)_{n=1}^{\infty}$ be a dense sequence in Y_1 , ③
 $(y''_n)_{n=1}^{\infty}$ be a dense sequence in Y_2 .
 Consider the open balls

$$U'_{nk} = \{y \in Y_1 : d(y, y'_n) < \frac{1}{k}\},$$

$$U''_{mk} = \{y \in Y_2 : d(y, y''_m) < \frac{1}{k}\}.$$

Then the $\mathcal{B}$ -algebra $\mathcal{B}(Y_1 \times Y_2)$ is generated already by the sets

$U'_{nk} \times U''_{mk}$ ($k, m, n \in \mathbb{N}$). In order to see this it suffices to show (recall Lecture 1) that $\forall (y_1, y_2) \in Y_1 \times Y_2$ and $\forall$ open set $U \subset Y_1 \times Y_2$ we can find k, m, n s.t.

$$(y_1, y_2) \in U'_{nk} \times U''_{mk} \subset U.$$

We consider on $Y_1 \times Y_2$ the metric

$$d((a_1, b_1), (a_2, b_2)) = d(a_1, a_2) + d(b_1, b_2).$$

Then we can find $\delta > 0$ s.t.

$$\{(z_1, z_2) : d(z_1, y_1) + d(z_2, y_2) < \delta\} \subset U.$$

Let $k \geq 1$ be s.t. $\frac{1}{k} < \frac{\delta}{4}$. We can find m, n s.t. $d(y_1, y'_n) < \frac{1}{k}$, $d(y_2, y''_m) < \frac{1}{k}$.

Then

$$(y_1, y_2) \in U'_{nk} \times U''_{nk} \subset U.$$

Indeed, if $(z_1, z_2) \in U'_{nk} \times U''_{nk}$, then

$$d(z_1, y_1) + d(z_2, y_2)$$

$$\leq d(z_1, y'_n) + d(y'_n, y_1) + d(z_2, y''_n) + d(y''_n, y_2)$$

$$< \frac{1}{k} + \frac{1}{k} + \frac{1}{k} + \frac{1}{k} = \frac{4}{k} < \delta.$$
