

Integration of positive functions

Assume $(X, \mathcal{B}, \mu)$ is a measure space.

Given a simple (measurable) function $f: X \rightarrow [0, +\infty)$, we can write

$$(*) \quad f = \sum_{k=1}^n c_k \mathbb{1}_{A_k}, \quad c_k \geq 0, \quad A_k \in \mathcal{B}.$$

We then define

$$\int_X f d\mu = \sum_{k=1}^n c_k \mu(A_k) \in [0, +\infty].$$

Lemma

$\int_X f d\mu$ is independent of the presentation of f as in $(*)$.

Proof

$$\text{Assume } f = \sum_{k=1}^n c_k \mathbb{1}_{A_k} = \sum_{k=1}^{n'} c'_k \mathbb{1}_{A'_k}$$

$(A_k, A'_k \in \mathcal{B}, c_k, c'_k \geq 0)$.

Consider the algebra of sets generated by A_k and A'_k , and let $B_1, \dots, B_m \in \mathcal{B}$ be its atoms.

We have $A_k = \bigcup_{i \in I_k} B_i$ (disjoint union)

for some $I_k \subset \{1, \dots, m\}$. Then

$$\sum_{k=1}^n c_k \mathbb{1}_{A_k} = \sum_{k=1}^n c_k \sum_{i \in I_k} \mathbb{1}_{B_i} = \sum_{i=1}^m \underbrace{\left(\sum_{k: i \in I_k} c_k \right)}_{\alpha_i} \mathbb{1}_{B_i}$$

The coefficients α_i depend only on f :

$$\alpha_i = f(x) \quad \forall x \in B_i.$$

Similarly we have

$$\begin{aligned}\sum_{k=1}^m c_k \mu(A_k) &= \sum_{k=1}^m c_k \sum_{i \in I_k} \mu(b_i) \\ &= \sum_{i=1}^m \left(\sum_{k: i \in I_k} c_k \right) \mu(b_i) \\ &= \sum_{i=1}^m d_i \mu(b_i).\end{aligned}$$

For the same reasons $\sum_{k=1}^m c'_k \mu(A'_k) = \sum_{i=1}^m d_i \mu(b_i)$
