

Integration of complex functions

①

Assume $(X, \mathcal{B}, \mu)$ is a measure space.

Def

A measurable function $f: X \rightarrow \mathbb{C}$ is called integrable (or μ -integrable) if

$$\int_X |f| d\mu < \infty.$$

Denote by $\mathcal{L}^1(X, \mathcal{B}, d\mu)$ or $\mathcal{L}^1(X, d\mu)$

the set of integrable functions

(also denoted by $\mathcal{L}^1_{\mathbb{C}}(\mu)$ in the book).

This is a vector space over $\mathbb{C}$, since

$$|f+g| \leq |f| + |g| \text{ and } |cf| = |c| |f| \quad (c \in \mathbb{C})$$

It is spanned by positive functions,

since

$$f = (\operatorname{Re} f)_+ - (\operatorname{Re} f)_- + i(\operatorname{Im} f)_+ - i(\operatorname{Im} f)_-,$$

where for a function h we let

$$h_+ = \max\{h, 0\}, \quad h_- = -\min\{h, 0\},$$

and if $f \in \mathcal{L}^1(X, d\mu)$, then

$$(\operatorname{Re} f)_\pm, (\operatorname{Im} f)_\pm \in \mathcal{L}^1(X, d\mu), \text{ as}$$

$$|(\operatorname{Re} f)_\pm|, |(\operatorname{Im} f)_\pm| \leq |f|.$$

Proposition

The integral extends uniquely from the positive integrable functions to a linear

functional $I'(X, d\mu) \rightarrow \mathbb{C}$, that is,

to a map such that

$$\int_X (f+g) d\mu = \int_X f d\mu + \int_X g d\mu, \quad \int_X cf d\mu = c \int_X f d\mu$$

($c \in \mathbb{C}$).

Proof

Uniqueness is clear, as positive functions in $I'(X, d\mu)$ span the entire space.

We first extend the integral to real integrable functions by letting

$$\int_X (g-h) d\mu \stackrel{\text{def}}{=} \int_X g d\mu - \int_X h d\mu$$

for $g, h \in I'(X, d\mu)$, $g, h \geq 0$.

This is well-defined, since if

$$g-h = g'-h',$$

then $g+h' = h+g'$ and hence $\int_X g d\mu + \int_X h' d\mu = \int_X g' d\mu + \int_X h d\mu$.

Then we extend the integral to the entire space $I'(X, d\mu)$ by

$$\int_X f d\mu \stackrel{\text{def}}{=} \int_X (\operatorname{Re} f) d\mu + i \int_X (\operatorname{Im} f) d\mu.$$

We easily get that by definition

$$\int_X (f_1 + f_2) d\mu = \int_X f_1 d\mu + \int_X f_2 d\mu \quad \forall f_1, f_2 \in I'(X, d\mu)$$

$$\text{and } \int_X cf d\mu = c \int_X f d\mu \quad \forall f \in I'(X, d\mu) \quad \forall c \geq 0$$

In order to ~~complete~~ prove the last

property for all $c \in \mathbb{C}$, it remains (2)
to check it for $c = -1$ and $c = i$.

For $c = -1$ it follows, since $g, h \geq 0$, then

$$\begin{aligned} \int_X (-(g-h)) d\mu &= \int_X (h-g) d\mu \\ &= \int_X h d\mu - \int_X g d\mu \\ &= - \int_X (g-h) d\mu. \end{aligned}$$

Similarly, for $c = i$ it is proved by a simple computation:

$$\begin{aligned} \int_X if d\mu &= \int_X \operatorname{Re}(if) d\mu + i \int_X \operatorname{Im}(if) d\mu \\ &= \int_X (-\operatorname{Im} f) d\mu + i \int_X (\operatorname{Re} f) d\mu \\ &= - \int_X (\operatorname{Im} f) d\mu + i \int_X (\operatorname{Re} f) d\mu \\ &= i \left(\int_X (\operatorname{Re} f) d\mu + i \int_X (\operatorname{Im} f) d\mu \right) \\ &= i \int_X f d\mu. \end{aligned}$$

Proposition ("triangle inequality")
For every $f \in L^1(X, d\mu)$, we have

$$\left| \int_X f d\mu \right| \leq \int_X |f| d\mu.$$

Proof

Choose $z \in \mathbb{T} \stackrel{\text{def}}{=} \{w \in \mathbb{C} : |w| = 1\}$

s.t. $z \int_X f d\mu \geq 0.$

Then

$$\left| \sum_X f d\mu \right| = \left| z \sum_X f d\mu \right| = z \sum_X f d\mu$$

$$= \sum_X z f d\mu$$

$$= \sum_X \operatorname{Re}(zf) d\mu + i \underbrace{\sum_X \operatorname{Im}(zf) d\mu}_{0''}$$

0'', since $\left| \sum_X f d\mu \right| \in \mathbb{R}$

$$= \sum_X (\operatorname{Re}(zf))_+ d\mu - \sum_X (\operatorname{Re}(zf))_- d\mu$$

$$\leq \sum_X (\operatorname{Re}(zf))_+ d\mu$$

$$\leq \sum_X |f| d\mu,$$

$$\text{since } (\operatorname{Re}(zf))_+ \leq |f|.$$

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