

completions of measure spaces

Def

A measure space $(X, \mathcal{B}, \mu)$ is called complete if whenever $A \in \mathcal{B}$, $\mu(A) = 0$, we have $B \in \mathcal{B}$ for all $B \subset A$.

Any measure space $(X, \mathcal{B}, \mu)$ can be completed as follows:

Let $\bar{\mathcal{B}}$ be the σ -algebra generated by $\mathcal{B}$ and all sets $B \subset X$ s.t. there exists $A \in \mathcal{B}$ with $B \subset A$ and $\mu(A) = 0$.

Proposition

The σ -algebra $\bar{\mathcal{B}}$ can also be described as follows:

$$\bar{\mathcal{B}} = \{ B \subset X : A_1 \subset B \subset A_2 \text{ for some } A_1, A_2 \in \mathcal{B} \text{ with } \mu(A_2 \setminus A_1) = 0 \}$$

with B, A_1, A_2 as above, we define

$$\bar{\mu}(B) \stackrel{\text{def}}{=} \mu(A_1) = \mu(A_2).$$

Then $(X, \bar{\mathcal{B}}, \bar{\mu})$ is a complete measure space.

Proof

Consider the collection of sets

$$\tilde{\mathcal{B}} = \{ B \subset X : A_1 \subset B \subset A_2 \text{ for some } A_1, A_2 \in \mathcal{B} \text{ with } \mu(A_2 \setminus A_1) = 0 \}.$$

This is a σ -algebra: $\emptyset \in \tilde{\mathcal{B}}$ (take $A_1 = A_2 = \emptyset$);
if $B \in \tilde{\mathcal{B}}$ and A_1, A_2 are as above, then

$A_2 \subset B \subset A_1$ and $\mu(A_1 \setminus A_2) = \mu(A_2 \setminus A_1) = 0$,
 so $B \in \tilde{\mathcal{B}}$. Finally, assume $(B_n)_{n=1}^{\infty} \subset \tilde{\mathcal{B}}$.
 Let $A'_n, A''_n \in \mathcal{B}$ be s.t.

$$A'_n \subset B_n \subset A''_n, \quad \mu(A''_n \setminus A'_n) = 0.$$

Put $A_1 = \bigcup_{n=1}^{\infty} A'_n$, $A_2 = \bigcup_{n=1}^{\infty} A''_n$. Then

$$A_1 \subset \bigcup_{n=1}^{\infty} B_n \subset A_2, \quad A_2 \setminus A_1 \subset \bigcup_{n=1}^{\infty} (A''_n \setminus A'_n),$$

so $\mu(A_2 \setminus A_1) \leq \sum_{n=1}^{\infty} \mu(A''_n \setminus A'_n) = 0$, hence $\bigcup_{n=1}^{\infty} B_n \in \tilde{\mathcal{B}}$.

Thus, $\tilde{\mathcal{B}}$ is a σ -algebra. It contains $\mathcal{B}$
 and all sets $B \subset X$ s.t. $B \subset A$ and $\mu(A) = 0$
 for some $A \in \mathcal{B}$. Hence $\overline{\mathcal{B}} \subset \tilde{\mathcal{B}}$. On the
 other hand, if $B \in \tilde{\mathcal{B}}$, $A_1 \subset B \subset A_2$,
 $A_1, A_2 \in \mathcal{B}$ and $\mu(A_2 \setminus A_1) = 0$, then
 $B = A_1 \cup (B \setminus A_1)$ and $B \setminus A_1 \subset A_2 \setminus A_1$.
 Hence $B \in \overline{\mathcal{B}}$. Thus $\overline{\mathcal{B}} = \tilde{\mathcal{B}}$.

Next, we need to show that $\bar{\mu}$ is

well-defined. Assume

$$A_1 \subset B \subset A_2, \quad A'_1 \subset B \subset A'_2,$$

with $A_1, A_2, A'_1, A'_2 \in \mathcal{B}$, $\mu(A_2 \setminus A_1) = \mu(A'_2 \setminus A'_1) = 0$.

Then $A_1 \setminus A'_1 \subset A_2 \setminus A'_1$, hence $\mu(A_1 \setminus A'_1) = 0$.

It follows that

$$\mu(A_1) = \mu(A_1 \cap A'_1).$$

For the same reason $\mu(A'_1) = \mu(A_1 \cap A'_1)$.

Therefore $\mu(A_1) = \mu(A'_1)$, so $\bar{\mu}$ is well-defined.

Finally, we have to check that μ is a measure. Assume $(B_n)_{n=1}^{\infty}$ is a sequence of disjoint sets in $\overline{B}$. As above, choose $A_n', A_n'' \in B$ s.t.

$$A_n' \subset B_n \subset A_n'' \text{ and } \mu(A_n'' \setminus A_n') = 0.$$

Then for $A_1 = \bigcup_{n=1}^{\infty} A_n'$ and $A_2 = \bigcup_{n=1}^{\infty} A_n''$ we have $A_1 \subset \bigcup_{n=1}^{\infty} B_n \subset A_2$ and $\mu(A_2 \setminus A_1) = 0$.

hence

$$\begin{aligned} \overline{\mu}\left(\bigcup_{n=1}^{\infty} B_n\right) &= \mu(A_1) = \mu\left(\bigcup_{n=1}^{\infty} A_n'\right) = \sum_{n=1}^{\infty} \mu(A_n') \\ &= \sum_{n=1}^{\infty} \overline{\mu}(B_n). \end{aligned}$$

We also have $\overline{\mu}(\emptyset) = \mu(\emptyset) = 0$. □

Def

If μ is a Borel measure on a metric space (X, d) , then the completion $\overline{B}(X)$ of the Borel σ -algebra with respect to μ is called the σ -algebra of μ -measurable sets.

For $\mu = \lambda_n$ on $\mathbb{R}^n$ we talk about the σ -algebra of Lebesgue measurable sets.

Instead of $\overline{\lambda}_n$ we still write λ_n and call it the Lebesgue measure. A function

$f: \mathbb{R}^n \rightarrow \mathbb{C}$ measurable with respect to the σ -algebra of Lebesgue measurable sets is called Lebesgue measurable.