

## $L_p$ -spaces

①

Assume  $V$  is a vector space over  $K = \mathbb{R}$  or  $K = \mathbb{C}$ .

Def

A seminorm on  $V$  is a map

$\rho: V \rightarrow [0, +\infty)$  s.t.

$$1) \rho(cx) = |c| \rho(x) \quad \forall x \in V \quad \forall c \in K;$$

$$2) \rho(x+y) \leq \rho(x) + \rho(y) \quad \forall x, y \in V$$

("triangle inequality").

A seminorm is called a norm if in addition

$$3) \rho(x) > 0 \quad \forall x \in V, x \neq 0.$$

A norm is usually denoted by  $\|\cdot\|$ .

A vector space with a norm is called a normed space.

Assume now that  $(X, \mathcal{B}, \mu)$  is a measure space. Fix  $1 \leq p < \infty$ . For a measurable function  $f: X \rightarrow \mathbb{C}$  define

$$\|f\|_p = \left( \int_X |f|^p d\mu \right)^{1/p} \in [0, +\infty].$$

Obviously  $\|cf\|_p = |c| \|f\|_p \quad \forall c \in \mathbb{C}$ .

Lemma

we have  $\|f+g\|_p \leq \|f\|_p + \|g\|_p$ .

Proof

If  $\|f\|_p = +\infty$  or  $\|g\|_p = +\infty$ , there is nothing to prove. If  $\|f\|_p < +\infty$ , then  $f = 0$  a.e.,

hence  $f+g = g$  a.e. and therefore

$\|f+g\|_p = \|g\|_p$ . Similarly, if  $\|g\|_p = 0$  then

$$\|f+g\|_p = \|f\|_p.$$

Assume now that  $0 < \|f\|_p, \|g\|_p < \infty$ .

$$\text{Put } \tilde{f} = \frac{f}{\|f\|_p}, \tilde{g} = \frac{g}{\|g\|_p}, t = \frac{\|f\|_p}{\|f\|_p + \|g\|_p}.$$

$$\text{Then } 0 < t < 1, \text{ and } \frac{f+g}{\|f\|_p + \|g\|_p} = t\tilde{f} + (1-t)\tilde{g}.$$

Observe that the function  $z \mapsto |z|^p$  is convex, so

$$|t\tilde{f}(x) + (1-t)\tilde{g}(x)|^p \leq t|\tilde{f}(x)|^p + (1-t)|\tilde{g}(x)|^p.$$

Integrating over  $X$  we get

$$\begin{aligned} \int_X |t\tilde{f} + (1-t)\tilde{g}|^p d\mu &\leq t \int_X |\tilde{f}|^p d\mu + (1-t) \int_X |\tilde{g}|^p d\mu \\ &= t + (1-t) = 1. \end{aligned}$$

$$\text{Therefore } \|t\tilde{f} + (1-t)\tilde{g}\|_p \leq 1.$$

$$\text{As } t\tilde{f} + (1-t)\tilde{g} = \frac{f+g}{\|f\|_p + \|g\|_p}, \text{ we see that}$$

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p. \quad \square$$

It follows that

$$\mathcal{L}^p(X, d\mu) \stackrel{\text{def}}{=} \left\{ f: X \rightarrow \mathbb{C} \mid \begin{array}{l} f \text{ is measurable,} \\ \|f\|_p < \infty \end{array} \right\}$$

is a vector space over  $\mathbb{C}$  with seminorm

$$f \mapsto \|f\|_p.$$

In general this is not a norm. (2)

Namely,  $\|f\|_p = 0$  if and only if  $f = 0$  a.e.

In general, if  $V$  is a vector space and  $q$  is a seminorm on  $V$ , then

$$V_0 = \{x \in V : q(x) = 0\}$$

is a subspace of  $V$  and on  $V/V_0$  we set a norm

$$\|x + V_0\| \stackrel{\text{def}}{=} q(x).$$

Therefore we get a normed space

$$L^p(X, d\mu) \stackrel{\text{def}}{=} L^p(X, d\mu) / \mathcal{N},$$

where  $\mathcal{N} = \{f: X \rightarrow \mathbb{R} \mid f \text{ is measurable, } f = 0 \text{ a.e.}\}$

The norm on  $L^p(X, d\mu)$  we continue to denote by  $\| \cdot \|_p$ . We often identify  $f \in L^p(X, d\mu)$  with the vector in  $L^p(X, d\mu)$  it represents.

Def A normed space  $(V, \| \cdot \|)$  is called a

Banach space if  $V$  is complete w.r.t. the metric  $d(x, y) = \|x - y\|$  on  $V$ .

### Theorem

Assume  $(X, \mathcal{B}, \mu)$  is a measure space,  $1 \leq p < \infty$ . Then  $L^p(X, d\mu)$  is a Banach space.

### Proof

Assume  $(f_n)_{n=1}^{\infty}$  is a Cauchy sequence in  $L^p(X, d\mu)$ . By passing to a subsequence we may assume that  $\|f_n - f_{n+1}\|_p < \frac{1}{2^n}$ .

Consider  $g_n = f_n - f_{n-1}$  (with  $g_1 = f_1$ ) and

$$G(x) = \sum_{n=1}^{\infty} |g_n(x)| \in [0, +\infty].$$

Note that for every  $N \geq 1$  we have

$$\begin{aligned} \left\| \sum_{n=1}^N |g_n| \right\|_p &\leq \sum_{n=1}^N \|g_n\|_p = \|f_1\|_p + \sum_{n=2}^N \|f_n - f_{n-1}\|_p \\ &< \|f_1\|_p + \sum_{n=2}^N \frac{1}{2^{n-1}} < \|f_1\|_p + 1, \end{aligned}$$

$$\text{so } \int_X \left( \sum_{n=1}^N |g_n| \right)^p d\mu \leq (\|f_1\|_p + 1)^p.$$

Letting  $N \rightarrow \infty$ , by the monotone convergence theorem we get

$$\int_X |G|^p d\mu \leq (\|f_1\|_p + 1)^p.$$

Hence  $G(x) < \infty$  for a.e.  $x$ . Define

$$f(x) = \begin{cases} \sum_{n=1}^{\infty} g_n(x), & \text{if } G(x) < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

As  $\sum_{n=1}^N g_n = f_N$ , we have  $f_N \rightarrow f$  a.e.

We also have  $|f| \leq G$  and  $|f_n| \leq G$ ,

so  $|f - f_n|^p \leq 2^p G^p$ . As  $\int_X G^p d\mu < \infty$ ,

by the dominated convergence theorem

we get

$$\int_X |f - f_n|^p d\mu \xrightarrow{n \rightarrow \infty} 0,$$

that is,  $f_n \rightarrow f$  in  $L^p(X, d\mu)$ .  $\square$

We can also take  $p = \infty$ . Assume for simplicity that  $(X, \mathcal{B}, \mu)$  is  $\sigma$ -finite. (3)

A measurable function  $f: X \rightarrow \mathbb{C}$  is called essentially bounded if

$|f| \leq C$  a.e. for some  $C \geq 0$ . The smallest such  $C$  is called the essential supremum of  $|f|$  and is denoted by  $\|f\|_\infty$ .

Then we define

$$L^\infty(X, \mu) = \{f: X \rightarrow \mathbb{C} \mid f \text{ is ess. bounded}\}$$

and

$$\|f\|_\infty = \frac{\|f\|_\infty}{1}$$

This is a normed space with norm  $\|\cdot\|_\infty$ .

Theorem

For every  $\sigma$ -finite measure space  $(X, \mathcal{B}, \mu)$ , the normed space  $L^\infty(X, \mu)$  is a Banach space.

Proof

Assume  $(f_n)_{n=1}^\infty$  is a Cauchy sequence in  $L^\infty(X, \mu)$ . Let

$$X_0 = \{x: |f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty \text{ for all } n, m \geq 1\}.$$

Then  $\mu(X \setminus X_0) = 0$  and for every  $x \in X_0$ ,  $(f_n(x))_{n=1}^\infty$  is a Cauchy sequence in  $\mathbb{C}$ . Let

$$f(x) = \begin{cases} \lim_{n \rightarrow \infty} f_n(x), & \text{if } x \in X_0, \\ 0, & \text{otherwise.} \end{cases}$$

Then for every  $x \in X_0$  we have

$$|f(x) - f_n(x)| = \lim_{m \rightarrow \infty} \|f_m(x) - f_n(x)\|$$

$$\leq \lim_{m \rightarrow \infty} \|f_m - f_n\|_\infty,$$

$$\text{so } \|f - f_n\|_\infty \leq \lim_{m \rightarrow \infty} \|f_m - f_n\|_\infty. \text{ Thus}$$

$$\lim_{n \rightarrow \infty} \|f - f_n\|_\infty \leq \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \|f_m - f_n\|_\infty = 0. \quad \square$$