

Dense subspaces of L^p

①

Theorem

Assume (X, d) is a metric space, μ is a Borel measure that is finite on every ball, $1 \leq p < \infty$. Then the space of bounded continuous functions with bounded support is dense in $L^p(X, d, \mu)$.

Bounded support means that f vanishes outside some ball.

Proof

We want to approximate $f \in L^p(X, d, \mu)$ by bounded continuous functions with bounded support. By considering separately $(\operatorname{Re} f)_\pm$ and $(\operatorname{Im} f)_\pm$ we may assume that $f \geq 0$. Then we can find simple functions f_n s.t. $0 \leq f_n \leq f$, $f_n \rightarrow f$ pointwise.

As $\|f - f_n\|_p \leq \|f\|_p$, by the dominated convergence theorem we have $f_n \rightarrow f$ in $L^p(X, d, \mu)$. Hence it suffices to consider simple f .

But then it suffices to approximate $f = \mathbb{1}_A$.

Note that $\mathbb{1}_A \in L^p(X, d, \mu)$ if and only if $\mu(A) < \infty$.

Fix $x_0 \in X$. Then $\mathbb{1}_{A \cap B_n(x_0)} \rightarrow \mathbb{1}_A$ pointwise, hence $\mathbb{1}_{A \cap B_n(x_0)} \rightarrow \mathbb{1}_A$ in $L^p(X, d, \mu)$, again by

the dominated convergence theorem.

Therefore it suffices to consider $A \subset B_n(x_0)$.

As μ is outer regular, we have

$$\mu(A) = \inf_{\substack{A \subset U \subset B_n(x_0) \\ U \text{ open}}} \mu(U)$$

Note that $\|\pi_U - \pi_A\|_p = \mu(U \setminus A)^{1/p}$.

Hence we can choose $U_k \subset B_n(x_0)$ st.
 $A \subset U_k$, U_k is open, $\pi_{U_k} \xrightarrow[k \rightarrow \infty]{} \pi_A$ in $L^p(X, d\mu)$.

Therefore it suffices to approximate π_U
for open $U \subset B_n(x_0)$. Consider the functions

$$f_k(x) = \frac{k d(x, U^c)}{1 + k d(x, U^c)}.$$

Then $0 \leq f_k \leq 1$, f_k is continuous,
supported on $\overline{U} \subset B_n(x_0)$ and $f_k \nearrow \pi_U$
pointwise, hence $f_k \xrightarrow[k \rightarrow \infty]{} \pi_U$ in $L^p(X, d\mu)$. $\square$

Modes of convergence

Assume $(X, \mathcal{B}, \mu)$ is a measure space.
Given measurable functions $f_n, f: X \rightarrow \mathbb{C}$,
recall that

$$f_n \rightarrow f \text{ a.e.}$$

means that $f_n(x) \rightarrow f(x)$ for all x
outside a set of measure zero.

Theorem (Egorov)

Assume $\mu(X) < \infty$ and $f_n \rightarrow f$ a.e. Then,
 $\forall \varepsilon > 0$, there exists $X_\varepsilon \in \mathcal{B}$ s.t.

$\mu(X_\varepsilon) < \varepsilon$ and $f_n \rightarrow f$ uniformly on $X \setminus X_\varepsilon$.

Proof

For $k, m \geq 1$, consider the set

$$X_{km} = \bigcup_{n=m}^{\infty} \{x: |f(x) - f_n(x)| > \frac{1}{k}\}.$$

For every k , the sequence $(X_{km})_{m=1}^{\infty}$ is
decreasing and

$$\bigcap_{m=1}^{\infty} X_{km} \subset \{x: f_n(x) \not\rightarrow f(x)\},$$

so $\mu(\bigcap_{m=1}^{\infty} X_{km}) = 0$. Hence $\mu(X_{km}) \downarrow 0$ as
 $m \rightarrow \infty$. For $\varepsilon > 0$ fixed, choose n_k s.t.

$$\mu(X_{kn_k}) < \frac{\varepsilon}{2^k}.$$

Put $X_\varepsilon = \bigcup_{k=1}^{\infty} X_{kn_k}$. Then $\mu(X_\varepsilon) < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon$.

If $x \notin X_\varepsilon$, then $x \notin X_{kn_k}$, so

$$|f(x) - f_n(x)| \leq \frac{1}{k} \text{ for all } n \geq n_k.$$

Therefore, $\sup_{x \in X \setminus X_\varepsilon} |f(x) - f_n(x)| \leq \frac{1}{K} \quad \forall n \geq n_K.$

hence

$$\lim_{n \rightarrow \infty} \sup_{x \in X \setminus X_\varepsilon} |f(x) - f_n(x)| = 0.$$

In addition to pointwise and uniform convergences we also consider the following:

$f_n \rightarrow f$ in the p-th mean if $\|f_n - f\|_p \xrightarrow{n \rightarrow \infty} 0$

($1 \leq p < \infty$) (for $p=1$, we say
in mean, for $p=2$ we say
in quadratic mean);

$f_n \rightarrow f$ in measure if $\forall \varepsilon > 0$ we have

$$\mu(\{x: |f(x) - f_n(x)| \geq \varepsilon\}) \xrightarrow{n \rightarrow \infty} 0.$$

Theorem

Assume $(X, \mathcal{B}, \mu)$ is a measure space,
 $1 \leq p < \infty$, $f_n, f: X \rightarrow \mathbb{C}$ are measurable functions.

Then

1) if $f_n \rightarrow f$ in the p-th mean, then

$f_n \rightarrow f$ in measure;

2) if $f_n \rightarrow f$ in measure, then there
is a subsequence $(f_{n_k})_{k=1}^\infty$ s.t. $f_{n_k} \rightarrow f$ a.e.

3) if $f_n \rightarrow f$ a.e. and $\mu(X) < \infty$, then

$f_n \rightarrow f$ in measure.

In particular, if $f_n \rightarrow f$ in the p-th mean,
then $f_{n_k} \rightarrow f$ a.e. for a subsequence $(f_{n_k})_k$.

Proof

1) If $f_n \rightarrow f$ in the p -th mean, then by Markov's inequality we have

$$\begin{aligned}\mu(\{x: |f(x) - f_n(x)| \geq \varepsilon\}) &\leq \frac{1}{\varepsilon^p} \int_X |f(x) - f_n(x)|^p d\mu(x) \\ &= \frac{1}{\varepsilon^p} \|f - f_n\|_p^p \xrightarrow{n \rightarrow \infty} 0.\end{aligned}$$

2) Assuming that $f_n \rightarrow f$ in measure, choose a subsequence $(f_{n_k})_{k=1}^\infty$ s.t. for

$$A_k = \{x: |f(x) - f_{n_k}(x)| \geq \frac{1}{2^k}\} \quad \text{we have}$$

$$\mu(A_k) < \frac{1}{2^k}. \quad \text{Consider the set}$$

$$A = \bigcap_{m=1}^\infty \bigcup_{k=m}^\infty A_k. \quad \text{Then}$$

$$\mu(A) \leq \mu\left(\bigcup_{k=m}^\infty A_k\right) \leq \sum_{k=m}^\infty \frac{1}{2^k} = \frac{1}{2^{m-1}},$$

$$\text{hence } \mu(A) = 0.$$

If $x \notin A$, then $x \notin \bigcup_{k=m}^\infty A_k$ for m large enough. Hence $|f(x) - f_{n_k}(x)| < \frac{1}{2^k}$ for $k \geq m$. Therefore $f_{n_k}(x) \rightarrow f(x)$ as $k \rightarrow \infty$.

3) If $f_n \rightarrow f$ a.e. and $\mu(X) < \infty$, then by Egorov's theorem, for every $\varepsilon > 0$, we can find X_ε s.t. $\mu(X_\varepsilon) < \varepsilon$ and $f_n \rightarrow f$ uniformly on $X \setminus X_\varepsilon$. Therefore, for $\varepsilon > 0$ and all n large enough,

we have $|f(x) - f_n(x)| < \delta \quad \forall x \in X \setminus X_\varepsilon$,
so

$$\mu(\{x: |f(x) - f_n(x)| \geq \delta\}) \leq \mu(X_\varepsilon) < \varepsilon.$$

It follows that $f_n \rightarrow f$ in measure. $\square$