

Hilbert spaces

①

Assume H is a vector space over $\mathbb{C}$.

Def

A pre-inner product on H is a map $(\cdot, \cdot): H \times H \rightarrow \mathbb{C}$ which is

1) sesquilinear, that is, linear in the first variable ~~and~~ and antilinear in the second:

$$(\alpha u + \beta v, w) = \alpha(u, w) + \beta(v, w),$$

$$(w, \alpha u + \beta v) = \bar{\alpha}(w, u) + \bar{\beta}(w, v)$$

for $u, v, w \in H$, $\alpha, \beta \in \mathbb{C}$;

2) Hermitian: $(u, v) = \overline{(v, u)}$;

3) positive semidefinite:

$$(u, u) \geq 0 \quad \text{for } u \in H.$$

It is called an inner product, or a scalar product, if instead of 3) the map is

3') positive definite:

$$(u, u) > 0 \quad \text{for } u \in H, u \neq 0.$$

A similar definition makes also sense for real vector spaces, but then

$(\cdot, \cdot): H \times H \rightarrow \mathbb{R}$ is

1) bilinear;

2) symmetric: $(u, v) = (v, u)$;

3) positive semidefinite (for pre-inner products) or

3') positive definite (for inner products).

We will only consider complex vector spaces, but all results will have analogues for real ones as well.

Cauchy-Schwarz inequality

If $(\cdot, \cdot)$ is a pre-inner product, then

$$|(u, v)| \leq (u, u)^{1/2} (v, v)^{1/2}$$

Proof

For every $t \in \mathbb{R}$ we have

$$0 \leq (u+tv, u+tv) = (u, u) + 2t \operatorname{Re}(v, u) + t^2 (v, v)$$

Now we use that for any quadratic polynomial $f(t) = at^2 + bt + c$ ($a, b, c \in \mathbb{R}$) we have

$$f \geq 0 \text{ on } \mathbb{R} \Leftrightarrow a \geq 0 \text{ and } b^2 - 4ac \leq 0.$$

$$\text{Thus, } |\operatorname{Re}(v, u)| \leq (u, u)^{1/2} (v, v)^{1/2}$$

We can find $z \in \mathbb{C}$ s.t. $z(u, v) \in \mathbb{R}$. Then

$$\begin{aligned} |(u, v)| &= |z(u, v)| = |\operatorname{Re}(zu, v)| \\ &\leq (zu, zu)^{1/2} (v, v)^{1/2} = (u, u)^{1/2} (v, v)^{1/2}. \end{aligned}$$

Corollary

we have a seminorm $\|u\| \stackrel{\text{def}}{=} (u, u)^{1/2}$.

It is a norm if and only if $(\cdot, \cdot)$ is an inner product.

Proof
we only need to check the triangle inequality:

$$\begin{aligned}\|u+v\|^2 &= (u+v, u+v) = \|u\|^2 + 2\operatorname{Re}(u, v) + \|v\|^2 \\ &\leq \|u\|^2 + 2\|u\| \cdot \|v\| + \|v\|^2 \\ &= (\|u\| + \|v\|)^2.\end{aligned}$$

Def
A Hilbert space is a complex vector space H with an inner product $(\cdot, \cdot)$ s.t. H is ~~an~~ complete with respect to the norm $\|u\| = (u, u)^{1/2}$.

The norm on a Hilbert space is determined by the inner product, but the inner product can also be recovered from the norm by the polarization identity:

$$(u, v) = \frac{\|u+v\|^2 - \|u-v\|^2}{4} + i \frac{\|u+iv\|^2 - \|u-iv\|^2}{4},$$

which is proved by a direct computation. A similar computation proves also the parallelogram law:

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2.$$

Fact
A norm on a vector space is given by an inner product if and only if it satisfies the parallelogram law, and

then the scalar product is uniquely determined by the polarization identity.

Example

Assume $(X, \mathcal{B}, \mu)$ is a measure space.

Then $L^2(X, d\mu)$ is a Hilbert space with inner product

$$(f, g) = \int_X f \bar{g} d\mu.$$

This is well-defined, as $|f \bar{g}| \leq \frac{1}{2}(1 + |f|^2 + |g|^2)$.

In particular, if $\mathcal{B} = \mathcal{P}(X)$ and μ is the counting measure, then $L^2(X, d\mu)$ is denoted by $\ell^2(X)$; for $X = \mathbb{N}$ we write simply ℓ^2 . Note that in this case for $f: X \rightarrow [0, +\infty]$ we have

$$\int_X f d\mu = \sum_{x \in X} f(x) \stackrel{\text{def}}{=} \sup_{\substack{F \subset X \\ F \text{ finite}}} \sum_{x \in F} f(x),$$

and if $\sum_{x \in X} f(x) < \infty$, then $\{x: f(x) > 0\}$ is at most countable, so $\sum_{x \in X} f(x) = \sum_{x: f(x) > 0} f(x)$ is the

Proof usual sum of a series.

For every finite $F \subset X$ we have

$$\int_X f d\mu \geq \int_F f d\mu = \sum_{x \in F} f(x).$$

$$\text{hence } \int_X f d\mu \geq \sup_{\substack{F \subset X \\ F \text{ finite}}} \sum_{x \in F} f(x) = \sum_{x \in X} f(x).$$

Therefore if $\int_X f d\mu < \infty$, then $\sum_{x \in X} f(x) < \infty$.

Then let $F_n = \{x: f(x) \geq \frac{1}{n}\}$.

We have $\sum_{x \in X} f(x) \geq \sum_{x \in F_n} \frac{1}{n}$, hence F_n is finite and therefore $\{x: f(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$ is countable. ③

Finally, if $f: X \rightarrow [0, +\infty]$ and $\{x: f(x) > 0\}$ is countable, so $\{x: f(x) > 0\} = \{x_n\}_{n=1}^N$

($N \in \mathbb{Z}_+ \cup \{+\infty\}$), then

$$\int_X f d\mu = \sum_{n=1}^N f(x_n),$$

for example, by the monotone convergence theorem. It is now easy to see that $\int_X f d\mu = \sum_{x \in X} f(x)$, since if $\{x_1, \dots, x_N\} \subset F$

and $F \cap \{x_n\}_{n=1}^N \subset \{x_1, \dots, x_{N+n}\}$, then

$$\sum_{n=1}^N f(x_n) \leq \sum_{x \in F} f(x) \leq \sum_{n=1}^{N+n} f(x_n). \quad \square$$

From the last part of the proof we see also that if $\sum_x f(x) < \infty$, then $\forall \varepsilon > 0$ there is finite $F_0 \subset X$ s.t. for every finite

$F \supset F_0$ we have $\left| \sum_{x \in X} f(x) - \sum_{x \in F} f(x) \right| < \varepsilon$.

Returning to $\ell^2(X)$, we obtain:

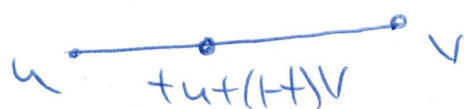
$$\ell^2(X) = \left\{ f: X \rightarrow \mathbb{C} \mid \sum_x |f(x)|^2 < \infty \right\}$$

is a Hilbert space with inner product

$$(f, g) = \sum_{x \in X} f(x) \overline{g(x)}, \text{ with the sum}$$

understood as in the previous paragraph.

Recall that a subset C of a vector space is called convex if $\forall u, v \in C \quad \forall t \in [0, 1]$ we have $tu + (1-t)v \in C$.



The following is one of the key properties of Hilbert spaces.

Theorem

Assume H is a Hilbert space, $C \subset H$ is a closed convex subset, $u \in H$. Then there is a unique $u_0 \in C$ s.t.

$$\|u - u_0\| = d(u, C) \quad (= \inf_{v \in C} \|u - v\|).$$

Proof

Let $d = d(u, C)$. Choose $u_n \in C$ s.t. $\|u - u_n\| \rightarrow d$. We claim that $(u_n)_{n=1}^\infty$ is a Cauchy sequence. As $\frac{u_n + u_m}{2} \in C$, we have

$$d^2 \leq \|u - \frac{u_n + u_m}{2}\|^2 = \frac{1}{4} \|(u - u_n) + (u - u_m)\|^2$$

parallelogram law

$$\frac{1}{4} (2\|u - u_n\|^2 + 2\|u - u_m\|^2 - \|u_n - u_m\|^2)$$

so

$$\|u_n - u_m\|^2 \leq \frac{2(\|u - u_n\|^2 - d^2) + 2(\|u - u_m\|^2 - d^2)}{2}$$

Thus, $(u_n)_{n=1}^\infty$ is indeed Cauchy, hence

$$u_n \rightarrow u_0 \in C \text{ for some } u_0 \text{ and}$$

$$\|u - u_0\| = \lim_{n \rightarrow \infty} \|u - u_n\| = d = d(u, C).$$

If u'_0 is another such point, we can take $u_{2n} = u_0$, $u_{2n+1} = u'_0$ and conclude that $u_0 = u'_0$.