

Orthogonal projections

①

For a Hilbert space H and a subset $A \subset H$,

Let $A^\perp = \{x \in H : x \perp y \text{ } \forall y \in A\},$

where $x \perp y$ means that $(x, y) = 0$. $A^\perp$ is a

Proposition closed subspace of H .

Assume H_0 is a closed subspace of a Hilbert space H . Then every $u \in H$ uniquely decomposes as

$$u = u_0 + u_1, \text{ with } u_0 \in H_0 \text{ and } u_1 \in H_0^\perp.$$

Moreover, $\|u - u_0\| = d(u, H_0)$ and $\|u\|^2 = \|u_0\|^2 + \|u_1\|^2$

Proof

Let $u_0 \in H_0$ be the unique point s.t.

$\|u - u_0\| = d(u, H_0)$. We need to show

that $u - u_0 \in H_0^\perp$. Take $v \in H_0$. Multiplying

v by a phase factor we may assume that

$(u - u_0, v) \in \mathbb{R}$. Consider the function f on $\mathbb{R}$

defined by

$$f(t) = \|u - u_0 - tv\|^2 = \|u - u_0\|^2 - 2t(u - u_0, v) + t^2\|v\|^2.$$

It attains its minimum at $t=0$, so

$f'(0) = 0$, that is, $(u - u_0, v) = 0$. Hence

we indeed have that $u_1 \stackrel{\text{def}}{=} u - u_0 \in H_0^\perp$.

Then $\|u\|^2 = \|u_0 + u_1\|^2 = \|u_0\|^2 + \|u_1\|^2$.

If $u = u'_0 + u'_1$ is another such decomposition, then $u_0 + u_1 = u'_0 + u'_1$,

so $u_0 - u'_0 = u'_1 - u_1$, hence $u_0 - u'_0 \in H_0 \cap H_0^\perp$

But then $u - u'_0 = 0$, as

$$\|u - u'_0\|^2 = (u - u'_0, u - u'_0) = 0.$$

Then also $u_1 = u'_1$.

For a closed subspace $H_0 \subset H$,
considers the map $P: H \rightarrow H_0$ s.t.
 $Pu \in H_0$ is the unique element satisfying
 $u - Pu \in H_0^\perp$.

The operator $P: H \rightarrow H_0$ is linear.
It is also contractive, meaning that
 $\|Pu\| \leq \|u\|$, since $\|u\|^2 = \|Pu\|^2 + \|u - Pu\|^2$.
It is called the orthogonal projection
onto H_0 .

If H_0 is finite dimensional with
an orthonormal basis $u_1, \dots, u_n$, then

$$Pu = \sum_{k=1}^n (u, u_k) u_k.$$

Orthonormal bases can be defined for
arbitrary Hilbert spaces.

Def
An orthonormal system in H is a
collection of vectors $u_i \in H$ ($i \in I$) s.t.
 $(u_i, u_j) = \delta_{ij} \quad \forall i, j \in I.$

It is called an orthonormal basis if

$\text{span}\{u_i\}_{i \in I}$ is dense in H .

Here $\text{span}\{u_i\}_{i \in I}$ denotes the linear
span of $\{u_i\}_{i \in I}$, the space of finite

Linear combinations of the vectors u_i . (2)

Theorem

Every Hilbert space H has an orthonormal basis. If H is separable, then there is a countable orthonormal basis.

Proof

Assume first that H is separable.

Let $(x_n)_{n=1}^{\infty}$ be a dense sequence in H .

By omitting all x_n s.t. $x_n \in \text{span}\{x_1, \dots, x_{n-1}\}$ we get a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ ($N \in \mathbb{N} \cup \{\infty\}$)

s.t. linearly independent vectors s.t.

$\text{span}\{x_{n_k}\}_{k=1}^{\infty} = \text{span}\{x_n\}_{n=1}^{\infty}$ is dense in H .

By applying the Gram-Schmidt procedure to $\{x_{n_k}\}_{k=1}^{\infty}$ we get an orthonormal basis

in H .

For general H we use Zorn's lemma to find a maximal orthonormal system $\{u_i\}_{i \in I}$ in H . If $\text{span}\{u_i\}_{i \in I} \neq H$, then there is a nonzero vector $u \in H$ orthogonal to $\text{span}\{u_i\}_{i \in I}$. Then $\{u_i\}_{i \in I} \cup \left\{ \frac{u}{\|u\|} \right\}$ is an orthonormal system, which contradicts maximality of $\{u_i\}_{i \in I}$. Hence $\{u_i\}_{i \in I}$ is an orthonormal basis. $\square$

Proposition

Assume $\{u_i\}_{i \in I}$ is an orthonormal system

in a Hilbert space H , Take $u \in H$. Then

- 1) $\sum_{i \in I} |(u, u_i)|^2 \leq \|u\|^2$ (Bessel's inequality);
in particular, $\{i: (u, u_i) \neq 0\}$ is countable;
- 2) if $\{u_i\}_{i \in I}$ is an orthonormal basis,
then
 $\sum_{i \in I} |(u, u_i)|^2 = \|u\|^2$ (Parseval's identity).

Proof

1) If $I \subset \mathbb{I}$ is finite, then for the orthogonal ~~projection~~ projection $P_F: H \rightarrow \text{span}\{u_i\}_{i \in F}$ we have

$$\begin{aligned} \|u\|^2 &\geq \|P_F u\|^2 = \left\| \sum_{i \in F} (u, u_i) u_i \right\|^2 \\ &= \sum_{i \in F} |(u, u_i)|^2. \end{aligned}$$

Hence $\sum_{i \in I} |(u, u_i)|^2 = \sup_{\substack{F \subset I \\ F \text{ finite}}} \sum_{i \in F} |(u, u_i)|^2 \leq \|u\|^2$

2) As $u \in \text{span}\{u_i\}_{i \in I}$, for every $\varepsilon > 0$ we can find a finite $F \subset I$ s.t.

$$d(u, \text{span}\{u_i\}_{i \in F}) < \varepsilon, \text{ that is,}$$

$$\|u - P_F u\| < \varepsilon. \text{ Then}$$

$$\begin{aligned} \sum_{i \in F} |(u, u_i)|^2 &= \|P_F u\|^2 = \|u\|^2 - \|u - P_F u\|^2 \\ &> \|u\|^2 - \varepsilon^2. \end{aligned}$$

Hence $\sum_{i \in I} |(u, u_i)|^2 > \|u\|^2 - \varepsilon^2$, and as $\varepsilon > 0$ was arbitrary, we get $\sum_{i \in I} |(u, u_i)|^2 \geq \|u\|^2$.



(3)
If $(u_i)_{i \in I}$ is an orthonormal basis,
then the numbers (u, u_i) ($i \in I$) are
called the Fourier coefficients of u

with respect to $(u_i)_{i \in I}$.

The Parseval identity suggests that u
is determined by its Fourier coefficients.
This is true, and even more, we have:

Proposition

Assume $(u_i)_{i \in I}$ is an orthonormal basis
in a Hilbert space H . Then for every vector
 $(c_i)_{i \in I} \in \ell^2(I)$ there is a unique
vector $u \in H$ with Fourier coefficients c_i .

We write $u = \sum_{i \in I} c_i u_i$.

Proof

The uniqueness follows from Parseval's
identity: if u, u' are two such vectors, then
 $u - u'$ has zero Fourier coefficients, hence
 $\|u - u'\| = 0$ and $u = u'$.

For the existence, note that as $\sum_{i \in I} |c_i|^2 < \infty$,
the set $\{i: c_i \neq 0\}$ is countable. Write

$$\{i: c_i \neq 0\} = \{i_n\}_{n=1}^N \quad (N \in \mathbb{N} \cup \{\infty\}).$$

If N is finite, we let $u = \sum_{n=1}^N c_{i_n} u_{i_n}$.

If $N = \infty$, then $\{v_n = \sum_{k=1}^n c_{i_k} u_{i_k}\}_{n=1}^{\infty}$ is

a Cauchy sequence, since if $n < m$, then

$$\|V_m - V_n\|^2 = \left\| \sum_{k=n+1}^m c_{i_k} u_{i_k} \right\|^2 = \sum_{k=n+1}^m |c_{i_k}|^2$$

$$\xrightarrow[n < m]{n, m \rightarrow \infty} 0, \text{ as } \sum_{k=1}^{\infty} |c_{i_k}|^2 < \infty.$$

Hence we can define $u = \lim_{n \rightarrow \infty} \sum_{k=1}^n c_{i_k} u_{i_k}$. $\square$

Remark

Equivalently, the element $u = \sum_{i \in I} c_i u_i$ can be described as the unique element in H s.t. $\forall \varepsilon > 0$ there is a finite $F_0 \subset I$ s.t.

$$\|u - \sum_{i \in F} c_i u_i\| < \varepsilon \quad \forall \text{ finite } F \supset F_0.$$

Corollary

We have a linear isomorphism

$$U: \ell^2(I) \xrightarrow{\sim} H, \quad U((c_i)_{i \in I}) = \sum_{i \in I} c_i u_i.$$

By Parseval's identity this isomorphism is isometric, that is, $\|Ux\| = \|x\| \quad \forall x \in \ell^2(I)$.
By the polarization identity this is equivalent to

$$(Ux, Uy) = (x, y) \quad \forall x, y \in \ell^2(I).$$

Therefore U is unitary.

Corollary

Up to a unitary isomorphism, there is only one infinite dimensional separable Hilbert space, namely, ℓ^2 .