

Given two orthonormal bases $(u_i)_{i \in I}$ and $(v_j)_{j \in J}$ in a Hilbert space H , we can decompose $u_i = \sum_{j \in J} (u_i, v_j) v_j$ and using that the sets $\{j : (u_i, v_j) \neq 0\}$ are countable prove the following

Fact

Any two orthonormal bases in a Hilbert space have the same cardinality.

Example (classical Fourier series)

Consider $H = L^2(0, 2\pi) = L^2([0, 2\pi], d\lambda)$

For $n \in \mathbb{Z}$, define $e_n(t) = \frac{1}{\sqrt{2\pi}} e^{int}$.

By a version of Weierstrass' theorem it is known that $\text{Span}\{e_n\}_{n \in \mathbb{Z}}$ is dense

in the supremum-norm in

$\{f \in C[0, 2\pi] : f(0) = f(2\pi)\}$.

As $C[0, 2\pi]$ is dense in $L^2(0, 2\pi)$, from this

one can deduce that $\text{Span}\{e_n\}_{n \in \mathbb{Z}}$ is dense in $L^2(0, 2\pi)$. We then see that $(e_n)_{n \in \mathbb{Z}}$ is an orthonormal basis in $L^2(0, 2\pi)$.

We therefore have a unitary isomorphism

$$\ell^2(\mathbb{Z}) \xrightarrow{\sim} L^2(0, 2\pi), (c_n)_{n \in \mathbb{Z}} \mapsto \sum_{n \in \mathbb{Z}} c_n e_n.$$

~~The~~ The Fourier coefficients of

$f \in L^2(0, 2\pi)$ are denoted by $\hat{f}(n)$, so

$$\hat{f}(n) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(t) e^{-int} dt$$

(More pedantically, $\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(t) e^{-int} d\lambda(t)$.)
Therefore we have $f = \sum_{n \in \mathbb{Z}} \hat{f}(n) e_n$ in $L^2(0, 2\pi)$

Fact (Carleson)

For every $f \in L^2(0, 2\pi)$, we have

$$\frac{1}{\sqrt{2\pi}} \sum_{n=-N}^N \hat{f}(n) e^{int} \xrightarrow{N \rightarrow \infty} f(t) \text{ for a.e. } t.$$

Dual spaces

Lemma

Assume V is a normed space over $K = \mathbb{R}$ or $K = \mathbb{C}$. ~~Consider~~ Consider a linear functional $f: V \rightarrow K$. TFAE:

- 1) f is continuous;
- 2) f is continuous at 0;
- 3) there is $C \geq 0$ s.t. $|f(x)| \leq C \|x\| \quad \forall x \in V$.

Proof

1) $\Rightarrow$ 2) is obvious

2) $\Rightarrow$ 3)

If f is continuous at 0, there is $R > 0$ s.t. $|f(x)| \leq 1$ whenever $\|x\| \leq R$.

For every $x \neq 0$ we then have

$$|f(x)| = \frac{\|x\|}{R} |f\left(\frac{R}{\|x\|} x\right)| \leq \frac{\|x\|}{R}.$$

3) $\Rightarrow$ 1)

If $|f(x)| \leq C \|x\|$ for all x , then

$$|f(x) - f(y)| = |f(x - y)| \leq C \|x - y\|,$$

hence f is continuous. Q

If 1)-3) are satisfied, then f is called a bounded linear functional. The smallest $C \geq 0$ in 3) is called the norm of f and denoted by $\|f\|$. We have

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} = \sup_{\|x\|=1} |f(x)| = \sup_{\|x\| \leq 1} |f(x)|.$$

Proposition

For every normed vector space V over $K = \mathbb{R}$ or $K = \mathbb{C}$, the bounded linear functionals on V form a Banach space V^* .

Proof

If $f, g : V \rightarrow K$ are bounded, then

$$\begin{aligned} |(f+g)(x)| &= |f(x) + g(x)| \leq |f(x)| + |g(x)| \\ &\leq \|f\| \cdot \|x\| + \|g\| \cdot \|x\|. \end{aligned}$$

Hence $f+g$ is bounded and $\|f+g\| \leq \|f\| + \|g\|$. Similarly, cf is bounded for every $c \in K$ and $\|cf\| = |c| \cdot \|f\|$.

Therefore $(V^*, \|\cdot\|)$ is a normed space.

We need to show that V^* is complete.

Assume $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence in V^* . As

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\| \cdot \|x\|, \quad (*)$$

$(f_n(x))_{n=1}^{\infty}$ is a Cauchy sequence in K for every $x \in V$. Define

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) \quad \text{for } x \in V.$$

Then f is a linear functional. Letting $m \rightarrow \infty$ in $(*)$, we get

$$|f_n(x) - f(x)| \leq \liminf_{m \rightarrow \infty} \|f_n - f_m\| \cdot \|x\|$$

hence $f_n - f$ is bounded, so f is bounded, and

$$\|f_n - f\| \leq \liminf_{m \rightarrow \infty} \|f_n - f_m\|.$$

But then

$$\lim_{n \rightarrow \infty} \|f_n - f\| \leq \lim_{n \rightarrow \infty} \liminf_{m \rightarrow \infty} \|f_n - f_m\| = 0,$$

so $f_n \rightarrow f$ in V^* . □

Remark

The sequence $\{\|f_n - f_m\|\}_{m=1}^{\infty}$ actually converges, since

$$|\|f_n - f_m\| - \|f_n - f_k\|| \leq \|f_m - f_k\|.$$

When we study/use normed spaces, it is often important to understand the dual spaces. For Hilbert spaces this is particularly easy:

Theorem (Riesz)

Assume H is a Hilbert space. Then every $f \in H^*$ has the form

$$f(x) = (x, y)$$

for a uniquely defined $y \in H$. Moreover, we have $\|f\| = \|y\|$.

Proof

The uniqueness is easy: if $f(x) = (x, y_1) = (x, y_2)$ for all x , then $(x, y_1 - y_2) = 0 \forall x$, and applying this to $x = y_1 - y_2$ we conclude that $y_1 = y_2$.

Let us prove the existence. If $f=0$, there is nothing to prove, as we can take $y=0$. Assume $f \neq 0$. Then $H_0 = \ker f$ is a proper closed subspace of H . Hence there $y_0 \in H_0^\perp$, $\|y_0\|=1$. Put

$$y = \overline{f(y_0)} y_0.$$

Then $y \in (\ker f)^\perp$ and $f(y) = |f(y_0)|^2 = (y, y)$.

For every $x \in H$, we have $x - \frac{f(x)}{f(y)} y \in \ker f$, so

$$(x, y) = (x - \frac{f(x)}{f(y)} y, y) + \frac{f(x)}{f(y)} (y, y) = f(x).$$

Finally, by the Cauchy-Schwarz inequality, $|f(x)| = |(x, y)| \leq \|x\| \cdot \|y\|$, so $\|f\| \leq \|y\|$. At the same time $f(y) = \|y\|^2$, so $\|f\| \geq \|y\|$. Thus, $\|f\| = \|y\|$. $\square$

For every Hilbert space H we can define the conjugate Hilbert space $\bar{H}$.

Its elements are the symbols $\bar{x}$ for $x \in H$. The linear structure and inner product are defined by

$$\bar{x} + \bar{y} = \overline{x+y}, \quad c \cdot \bar{x} = \overline{\bar{c} x}, \quad (\bar{x}, \bar{y}) = \overline{(x, y)} = (y, x).$$

Corollary

~~For~~ For every Hilbert space H , we have an ~~isometric~~ isomorphism $\bar{H} \cong H^*$,

$$\bar{x} \mapsto (\cdot, x).$$