

Hahn-Banach theorem

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Theorem (Hahn-Banach)

Assume V is a real vector space, $V_0 \subset V$ a subspace, $\varphi: V \rightarrow \mathbb{R}$ a convex function and $f: V_0 \rightarrow \mathbb{R}$ a linear functional s.t. $f \leq \varphi$ on V_0 . Then f can be extended to a linear functional F on V s.t. $F \leq \varphi$.

Proof

Assume first that $V = V_0 + \mathbb{R}x$ for some $x \in V \setminus V_0$. To define F we need to specify $F(x)$. Condition $F \leq \varphi$ means that we need

$$F(y \pm tx) \leq \varphi(y \pm tx) \quad \forall y \in V_0 \quad \forall t > 0,$$

that is,

$$\pm t F(x) + f(y) \leq \varphi(y \pm tx).$$

Dividing by t , this is equivalent to the inequalities

$$\begin{cases} F(x) \leq \frac{\varphi(y+tx) - f(y)}{t}, \\ F(x) \geq \frac{f(y) - \varphi(y-tx)}{t}, \end{cases}$$

$\forall y \in V_0 \quad \forall t > 0$. To show that there is a number satisfying these inequalities, we need to check that

$$\sup_{\substack{y \in V_0 \\ t > 0}} \frac{f(y) - \varphi(y-tx)}{t} \leq \inf_{\substack{y \in V_0 \\ t > 0}} \frac{\varphi(y+tx) - f(y)}{t}.$$

Then as $F(x)$ we can take any number in the interval $[s, t]$. In other words, we need to check that

$$\frac{f(y) - \varphi(y - tx)}{t} \leq \frac{\varphi(z + sx) - f(z)}{s}$$

for all $y, t \in V_0$ and $s, t > 0$. Equivalently,

$$sf(y) + t f(z) \leq s \varphi(y - tx) + t \varphi(z + sx).$$

We have:

~~$$sf(y) + t f(z) = f\left(\frac{s}{s+t}y + \frac{t}{s+t}z\right)$$~~

$$\frac{sf(y) + t f(z)}{s+t} = f\left(\frac{s}{s+t}y + \frac{t}{s+t}z\right)$$

$$\leq \varphi\left(\frac{s}{s+t}y + \frac{t}{s+t}z\right)$$

$$= \varphi\left(\frac{s}{s+t}(y - tx) + \frac{t}{s+t}(z + sx)\right)$$

$$\leq \frac{s}{s+t} \varphi(y - tx) + \frac{t}{s+t} \varphi(z + sx),$$

which is what we need.

The general case is deduced from this using transfinite induction (Zorn's lemma), as follows. Consider the set X of pairs (W, F) , where $W \subset V$ is a subspace, $F: W \rightarrow \mathbb{R}$ a linear functional, $V_0 \subset W$ and $F|_{V_0} = f$. Define a partial order on X by $F \leq \varphi$ on W

$(W_1, F_1) \leq (W_2, F_2)$ iff $W_1 \subset W_2$ and $F_1 = F_2|_{W_1}$.

If C is a chain in X , then it has an upper bound (W, F) defined by

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$$W = \bigcup_{(Y, g) \in C} Y, \quad F(y) = g(y) \text{ if } y \in Y, (Y, g) \in C.$$

Hence X has a maximal element (W, F) by Zorn's lemma. If $W \neq V$, then we can take $x \in V \setminus W$ and extend F to $W + \mathbb{R}x$ preserving the inequality $F \leq \varphi$. This contradicts maximality of (W, F) in X . Hence $W = V$. $\square$

Theorem (Hahn-Banach)

Assume V is a real or complex vector space, p a seminorm on V , $V_0 \subset V$ a subspace and f a linear functional on V_0 s.t. $|f(x)| \leq p(x) \forall x \in V_0$. Then f can be extended to a linear functional F on V s.t. $|F(x)| \leq p(x) \forall x \in V$.

Proof

In the real case we apply the previous theorem to $p = p$. Then we get a linear functional F on V s.t.

$$F|_{V_0} = f, \quad F(x) \leq p(x) \forall x \in V.$$

But then $-F(x) = F(-x) \leq p(-x) = p(x)$, hence $|F(x)| \leq p(x)$.

Consider now the complex case.

We can view V_0 and V as real vector spaces. We can then extend the real linear functional $g = \operatorname{Re} f : V_0 \rightarrow \mathbb{R}$ to a real linear functional $G : V \rightarrow \mathbb{R}$ s.t. $|G(x)| \leq p(x)$.

Define $F : V \rightarrow \mathbb{C}$ by

$$F(x) = G(x) - i G(ix).$$

Then F is a real-linear ~~functional~~ ^{map $V \rightarrow \mathbb{C}$} and $F(ix) = iF(x)$, hence F is a complex linear functional. As $\operatorname{Im} f(x) = -\operatorname{Re} f(ix) = -g(ix)$, F extends f .

It remains to check that $|F(x)| \leq p(x)$ for all $x \in V$. Given x , we can find $z \in \mathbb{T}$ s.t. $zF(x) \in \mathbb{R}$. Then $zF(x) = F(zx) = G(zx)$, so $|F(x)| = |zF(x)| = |F(zx)| = |G(zx)| \leq p(zx) = p(x)$. $\square$

Corollary

Assume V is a normed vector space (real or complex), $V_0 \subset V$ a subspace and $f \in V_0^*$. Then there is $F \in V^*$ s.t.

$$F|_{V_0} = f \quad \text{and} \quad \|F\| = \|f\|.$$

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Proof

We apply the previous theorem to $P(x) = \|f\| \cdot \|x\|$.

□

Corollary

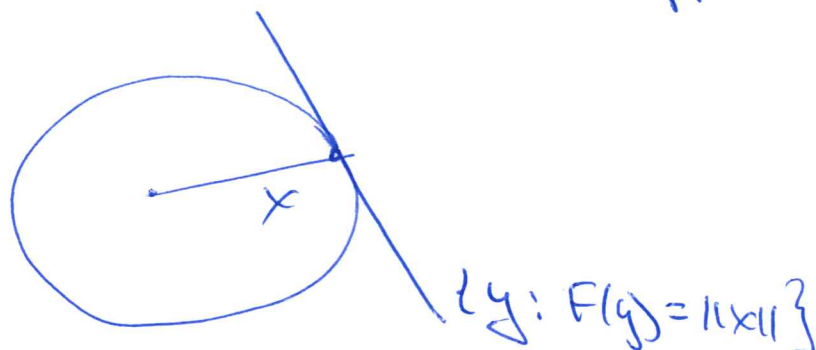
Assume V is a normed space and $x \in V, x \neq 0$. Then there is $F \in V^*$ s.t. $\|F\| = 1$ and $F(x) = \|x\|$.

Proof

We apply the previous corollary to $V_0 = K \times (K = \mathbb{R}, \mathbb{C})$ and $f: V_0 \rightarrow K$ defined by $f(cx) = c\|x\|$.

□

Such an F is called a supporting functional at x .



If V is a normed vector space, then every $x \in V$ defines a bounded linear functional on V^* by $V^* \ni f \mapsto f(x)$.

As $|f(x)| \leq \|f\| \cdot \|x\|$, this functional has norm $\leq \|x\|$. By using a supporting functional at x , we actually see that it we get norm $\|x\|$. Thus, we have an isometric

embedding $V \hookrightarrow V^{**} \stackrel{\text{Id}}{=} (V^*)^*$.

We can therefore view V as a subspace of V^{**} .

Def

A normed space V is called reflexive if $V^{**} = V$.

Remark/warning

This is stronger than requiring $V \simeq V^{**}$.

Examples

1) Every f.d. normed vector space V is reflexive for dimension reasons:

$$\dim V^{**} = \dim V^* = \dim V.$$

2) Every Hilbert space H is reflexive.

Indeed, $H^* = \overline{H}$. By Riesz' theorem every bounded linear functional f on $\overline{H}$ has the form

$$f(x) = (x, \bar{y}) = (y, x) \text{ for some } y \in H,$$

which exactly means that $f = y$ in H^{**} .

As we will see later, the spaces

$L^p(X, d\mu)$, with μ σ -finite and $1 < p < \infty$, are reflexive. The spaces $L^1(X, d\mu)$ and

$L^\infty(X, \mu)$ are usually not reflexive.