

Radon-Nikodym theorem

①

Assume $(X, \mathcal{B}, \mu)$ is a measure space.
Are there other measures on $(X, \mathcal{B})$?

Example

Take a measurable function $f: X \rightarrow [0, +\infty]$
and define

$$\nu(A) = \int_A f d\mu \quad \text{for } A \in \mathcal{B}.$$

This is a measure by the monotone
convergence theorem. We write
 $d\nu = f d\mu$.

Proposition

Assume $(X, \mathcal{B})$ is a measurable space,
 μ and ν are σ -finite measures on $(X, \mathcal{B})$.
Then there exist $N \in \mathcal{B}$ and a measurable
 $f: X \rightarrow [0, +\infty)$ s.t.

$$\mu(N) = 0 \quad \text{and} \quad \nu(A) = \nu(A \cap N) + \int_A f d\mu \quad \forall A \in \mathcal{B}.$$

Proof

Assume first that the measures μ, ν are
finite. Consider the measure $\eta = \mu + \nu$
and define a linear functional φ on
 $L^2(X, d\eta)$ by

$$\varphi(g) = \int_X g d\nu.$$

It is well-defined and bounded, since

$$|\varphi(g)| \leq \int_X |g| d\nu \leq \left(\int_X |g|^2 d\nu \right)^{1/2} \nu(X)^{1/2}$$

Cauchy-Schwarz

$$\leq \nu(X)^{1/2} \left(\int_X |g|^2 d\eta \right)^{1/2}.$$

By Riesz' theorem there exists $h \in L^2(X, d\eta)$ s.t.
 $\psi(g) = \int_X gh d\eta$ for all $g \in L^2(X, d\eta)$.

For $g = \pi_A$ we get

$$\nu(A) = \int_A h d\eta \quad \forall A \in \mathcal{B}.$$

In particular, $h \geq 0$ (η -a.e.). As $\nu(A) \leq \eta(A)$, we also have $h \leq 1$ (η -a.e.). From now on we view h as a function on X and assume $0 \leq h \leq 1$.

For $g = \pi_A$ ($A \in \mathcal{B}$) we have

$$\begin{aligned} \int_X g d\nu &= \int_X \psi(g) d\eta = \int_X gh d\eta \\ &= \int_X gh d\mu + \int_X gh d\nu, \text{ hence} \\ \int_X g(1-h) d\nu &= \int_X gh d\mu \quad (*) \end{aligned}$$

By extending (*) to positive simple functions and then using the monotone convergence theorem, we conclude that (*) holds for all measurable $g: X \rightarrow [0, +\infty]$. We now let

$$N = \{x: h(x) = 1\} \text{ and } f = \frac{h}{1-h} \pi_{N^c}.$$

Letting $g = \pi_N$ in (*) we get $0 = \mu(N)$.

For $A \in \mathcal{B}$, letting $g = \frac{\pi_{A \cap N^c}}{1-h}$ in (*)

we get

$$\begin{aligned} \nu(A \cap N^c) &= \int_X \frac{\pi_A \pi_{N^c}}{1-h} h d\mu \\ &= \int_X \pi_A f d\mu = \int_A f d\mu. \end{aligned}$$

Thus

$$\begin{aligned} \nu(A) &= \nu(A \cap N) + \nu(A \cap N^c) \\ &= \nu(A \cap N) + \int_A f d\mu. \end{aligned}$$

This finishes the proof for finite μ and ν .

If μ and ν are σ -finite, we can write X as disjoint unions $X = \bigcup_{n=1}^{\infty} X_n = \bigcup_{m=1}^{\infty} Y_m$

with $X_n, Y_m \in \mathcal{B}$, $\mu(X_n) < \infty$, $\nu(Y_m) < \infty$.

Applying the first part of the proof to $X_n \cap Y_m$, we find $N_{nm} \in \mathcal{B}$, $N_{nm} \subset X_n \cap Y_m$, and measurable $f_{nm}: X_n \cap Y_m \rightarrow [0, +\infty)$

s.t.

$$\begin{aligned} \mu(N_{nm}) &= 0 \text{ and } \nu(A \cap X_n \cap Y_m) \\ &= \nu(A \cap N_{nm}) + \int_{A \cap X_n \cap Y_m} f_{nm} d\mu. \end{aligned}$$

We then put $N = \bigcup_{n,m=1}^{\infty} N_{nm}$ and

define $f: X \rightarrow [0, +\infty)$ by letting $f = f_{nm}$ on $X_n \cap Y_m$. □

When can we discard the term $\nu(A \cap N)$?

Def

Given measures μ and ν on $(X, \mathcal{B})$, we say that ν is absolutely continuous with respect to μ and write $\nu \ll \mu$, if $\nu(A) = 0$ whenever $A \in \mathcal{B}$, $\mu(A) = 0$.

The name is partly explained by the following result.

Lemma

Assume μ and ν are measures on $(X, \mathcal{B})$, $\nu(X) < \infty$. Then $\nu \ll \mu$ if and only if $\forall \varepsilon > 0 \exists \delta > 0$ s.t. if $A \in \mathcal{B}$, $\mu(A) < \delta$, then $\nu(A) < \varepsilon$.

Proof

($\Leftarrow$) Obvious

($\Rightarrow$) Assume this is not true. Hence there is $\varepsilon > 0$ s.t. $\forall \delta > 0$ we can find $A \in \mathcal{B}$ satisfying $\mu(A) < \delta$, $\nu(A) \geq \varepsilon$.
Let A_n be such a set A for $\delta = \frac{1}{2^n}$.

Put $A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$. Then

$$\begin{aligned} \mu(A) &\leq \lim_{n \rightarrow \infty} \mu\left(\bigcup_{k=n}^{\infty} A_k\right) \leq \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} \mu(A_k) \\ &\leq \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} \frac{1}{2^k} = \lim_{n \rightarrow \infty} \frac{1}{2^{n-1}} = 0. \end{aligned}$$

As $\nu(X) < \infty$, we also have

$$\nu(A) = \lim_{n \rightarrow \infty} \nu\left(\bigcup_{k=n}^{\infty} A_k\right) \geq \varepsilon.$$

This contradicts the assumption $\nu \ll \mu$. $\square$

Remark

The result is not true for infinite ν .

Theorem (Radon-Nikodym)

③

Assume μ and ν are σ -finite measures on a measurable space $(X, \mathcal{B})$, $\nu \ll \mu$.

Then there is a measurable function $f: X \rightarrow \{0, +\infty\}$ s.t. $d\nu = f d\mu$

(that is, $\nu(A) = \int_A f d\mu$). If $\tilde{f}$ is another function with the same properties, then $f = \tilde{f}$ μ -a.e.

The function f is called the Radon-Nikodym derivative of ν w.r.t. μ and is denoted by $\frac{d\nu}{d\mu}$.

Proof

By the proposition above, we can find

$N \in \mathcal{B}$ and $f: X \rightarrow \{0, +\infty\}$ s.t.

$$\mu(N) = 0, \quad \nu(A) = \nu(A \cap N) + \int_A f d\mu.$$

As $\nu(A \cap N) = 0$ by the assumption $\nu \ll \mu$,

this proves the existence of f .

Assume we have another $\tilde{f}$. Then

$$\nu(A) = \int_A f d\mu = \int_A \tilde{f} d\mu \quad \forall A \in \mathcal{B}.$$

If $B \in \mathcal{B}$, $\nu(B) < \infty$, then consider

$$A_1 = \{x \in B: f(x) > \tilde{f}(x)\}, \quad A_2 = \{x \in B: f(x) < \tilde{f}(x)\}$$

Then $\int_{A_1} (f - \tilde{f}) d\mu = 0$ and $\int_{A_2} (\tilde{f} - f) d\mu = 0$,

hence $\mu(A_1) = \mu(A_2) = 0$. Therefore

$f = \tilde{f}$ μ -a.e. on B . As γ is σ -finite, we have $X = \bigcup_{n=1}^{\infty} B_n$, $\gamma(B_n) < \infty$.

Then $f = \tilde{f}$ μ -a.e. on B_n for all n , hence $f = \tilde{f}$ μ -a.e. on X . $\square$

Example

Consider a real-valued function $f \in C^1[a, b]$ s.t. $f'(t) > 0 \forall t \in [a, b]$. Let $c = f(a)$, $d = f(b)$. We know that for any Riemann integrable function g on $[c, d]$ we have

$$\int_c^d g(t) dt = \int_a^b g(f(t)) f'(t) dt.$$

Equivalently, for any Riemann integrable function g on $[a, b]$ we have

$$\int_c^d g \circ f^{-1} dt = \int_a^b g f' dt. \quad (*)$$

Denote by $\lambda_{[a, b]}$, $\lambda_{[c, d]}$ the Lebesgue measure restricted to the Borel subsets of $[a, b]$ and $[c, d]$, respectively. Then (*) implies that

$$d((f^{-1})_* \lambda_{[c, d]}) = f' d\lambda_{[a, b]},$$

since the integration of $g = \mathbb{1}_{[\alpha, \beta]}$ gives the same results for any interval $[\alpha, \beta] \subset [c, d]$ and since a finite Borel measure on $[a, b]$ is determined by its values on such intervals.

Thus, $(f^{-1})_* \lambda_{[c, d]} \ll \lambda_{[a, b]}$ and

$$\frac{d((f^{-1})_* \lambda_{[c, d]})}{d\lambda_{[a, b]}} = f'$$