

Decomposition theorems

①

Def

Two measures ν and μ on a measurable space $(X, \mathcal{B})$ are called mutually singular, or we say that ν is singular w.r.t. μ , if there is $N \in \mathcal{B}$ s.t. $\nu(N^c) = 0, \mu(N) = 0$. We then write $\nu \perp \mu$.

Theorem (Lebesgue decomposition theorem)

Assume ν, μ are σ -finite measures on $(X, \mathcal{B})$. Then there exist unique measures ν_a and ν_s s.t.

$$\nu = \nu_a + \nu_s, \quad \nu_a \ll \mu, \quad \nu_s \perp \mu.$$

Proof

As we showed earlier, there exist $N \in \mathcal{B}$ and measurable $f: X \rightarrow [0, +\infty)$ s.t. $\mu(N) = 0$ and

$$\nu(A) = \nu(A \cap N) + \int_A f d\mu \quad \forall A \in \mathcal{B}.$$

We then define

$$\nu_a(A) = \int_A f d\mu, \quad \nu_s(A) = \nu(A \cap N).$$

Assume we have another such decomposition $\nu = \tilde{\nu}_a + \tilde{\nu}_s$. Let $\tilde{N} \in \mathcal{B}$

be s.t. $\mu(\tilde{N}) = 0$ and $\tilde{\nu}_s(\tilde{N}^c) = 0$.

Then $\mu(N \cup \tilde{N}) = 0$ and $\nu_s((N \cup \tilde{N})^c) = 0$, $\tilde{\nu}_s((N \cup \tilde{N})^c) = 0$. Hence, for every $A \in \mathcal{B}$,

$$\begin{aligned}
 \nu_s(A) &= \nu_s(A \cap (N \cup \tilde{N})) \\
 &= \nu_s(A \cap (N \cup \tilde{N})) + \nu_a(A \cap (N \cup \tilde{N})) \\
 &= \nu(A \cap (N \cup \tilde{N})),
 \end{aligned}$$

and for the same reasons

$$\tilde{\nu}_s(A) = \nu(A \cap (N \cup \tilde{N})).$$

Thus, $\nu_s = \tilde{\nu}_s$.

In a similar way,

$$\begin{aligned}
 \nu_a(A) &= \nu_a(A \cap (N \cup \tilde{N})^c) \\
 &= \nu_a(A \cap (N \cup \tilde{N})^c) + \nu_s(A \cap (N \cup \tilde{N})^c) \\
 &= \nu(A \cap (N \cup \tilde{N})^c),
 \end{aligned}$$

$$\tilde{\nu}_a(A) = \nu(A \cap (N \cup \tilde{N})^c).$$

hence $\nu_a = \tilde{\nu}_a$. □

Theorem (polar decomposition of complex measures)

Assume ν is a complex measure on $(X, \mathcal{B})$. Then there exist a finite measure μ on $(X, \mathcal{B})$ and a measurable function $f: X \rightarrow \mathbb{C}$ s.t. $d\nu = f d\mu$. If $(\tilde{\mu}, \tilde{f})$ is another such pair, then $\tilde{\mu} = \mu$ and $\tilde{f} = f$ μ -a.e.

Proof

Take $\mu = |\nu|$. Then $\nu \ll |\nu|$.

hence by the Radon-Nikodym theorem

for complex measures, we have $d\nu = f d\mu$ for a unique $f \in L^1(X, d\mu)$. ②

Then $d|\nu| = |f| d\mu = |f| d|\nu|$.

Hence $|f| = 1$ (μ -a.e.). By viewing f as a function, we can therefore assume that $f: X \rightarrow \mathbb{T}$.

Assume we have another such decomposition $d\nu = \tilde{f} d\tilde{\mu}$. Then $d|\nu| = |\tilde{f}| d\tilde{\mu} = d\tilde{\mu}$, that is, $\tilde{\mu} = \mu = |\nu|$. Then $d\nu = \tilde{f} d\mu$, hence $\tilde{f} = f$ μ -a.e. □

Duals of L^p -spaces

(1)

Assume $(X, \mathcal{B}, \mu)$ is a measure space, $1 \leq p < \infty$.
What is the dual of $L^p(X, \mathcal{B}, \mu)$?

When does a measurable function $g: X \rightarrow \mathbb{C}$
define a bounded linear functional on $L^p(X, \mathcal{B}, \mu)$ by
$$\varphi(f) = \int_X fg d\mu$$
?

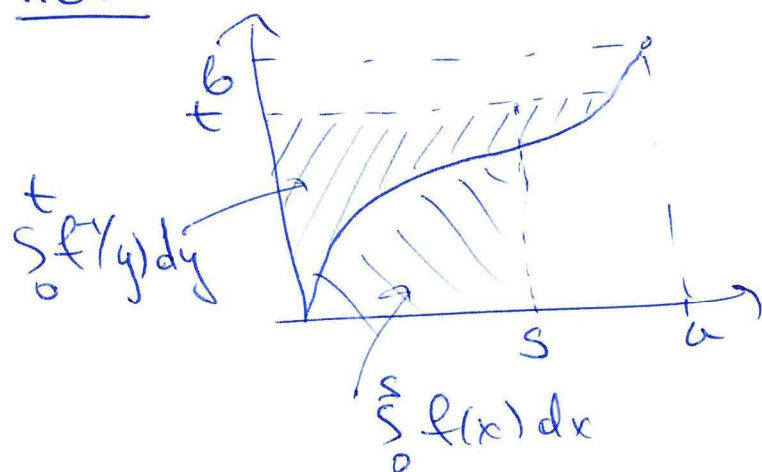
Young's inequality

Assume $f: [0, a] \rightarrow [0, b]$ is a strictly increasing
continuous function, $f(0)=0$, $f(a)=b$. Then,
for all $s \in [0, a]$ and $t \in [0, b]$, we have

$$st \leq \int_0^s f(x) dx + \int_0^t f^{-1}(y) dy,$$

and the equality holds if and only if $t = f(s)$.

Proof



Compare areas

Apply this to $f(s) = s^{p-1}$. Then $f^{-1}(t) = t^{q-1}$,
where q is the Hölder conjugate of p :

$(p-1)(q-1)=1$, that is, $\frac{1}{p} + \frac{1}{q} = 1$. We get

$$st \leq \int_0^s x^{p-1} dx + \int_0^t y^{q-1} dy = \frac{s^p}{p} + \frac{t^q}{q}.$$

Hölder inequality

If $f \in L^p(X, d\mu)$, $g \in L^q(X, d\mu)$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$,
then $fg \in L^1(X, d\mu)$ and
$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

Proof

Assume $\|f\|_p > 0$, $\|g\|_q > 0$, otherwise $fg = 0$ (a.e.)
we have

$$\frac{|fg|}{\|f\|_p \|g\|_q} \leq \frac{1}{p} \frac{|f|^p}{\|f\|_p^p} + \frac{1}{q} \cdot \frac{|g|^q}{\|g\|_q^q}$$

Integrating over X we get

$$\begin{aligned} \frac{\|fg\|_1}{\|f\|_p \|g\|_q} &\leq \frac{1}{p} \int_X \frac{|f|^p}{\|f\|_p^p} d\mu + \frac{1}{q} \int_X \frac{|g|^q}{\|g\|_q^q} d\mu \\ &= \frac{1}{p} \frac{\|f\|_p^p}{\|f\|_p^p} + \frac{1}{q} \frac{\|g\|_q^q}{\|g\|_q^q} \\ &= \frac{1}{p} + \frac{1}{q} = 1, \end{aligned}$$

so $\|fg\|_1 \leq \|f\|_p \|g\|_q$. □

It follows that every $g \in L^q(X, d\mu)$
defines a bounded linear functional

$$\ell_g : L^p(X, d\mu) \rightarrow \mathbb{C}, \quad \ell_g(f) = \int_X fg d\mu,$$

and $\|\ell_g\| \leq \|g\|_q$.

The same makes sense for $p=1, q=\infty$
and $p=\infty, q=1$, when μ is σ -finite,

as
$$\int_X |fg| d\mu \leq \int_X |f| d\mu \|g\|_\infty = \|f\|_1 \|g\|_\infty$$

(2)

Lemma

Assume $1 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and μ is σ -finite if $p=1$ or $p=\infty$. For $g \in L^q(X, d\mu)$ consider $\ell_g \in L^p(X, d\mu)^*$. Then

$$\|\ell_g\| = \|g\|_q.$$
Proof

We may assume $\|g\|_q > 0$, otherwise $\ell_g = 0$. Consider three cases:

1) $1 < p < \infty$, so $1 < q < \infty$.

We already know that $\|\ell_g\| \leq \|g\|_q$.

To prove the opposite inequality, consider

$$f = \frac{\overline{g}}{|g|} |g|^{q-1}. \text{ Then}$$

$$|f|^p = |g|^{(q-1)p} = |g|^{q(1-\frac{1}{q})p} = |g|^q,$$

so $\|f\|_p^p = \|g\|_q^q$. We have

$$\ell_g(f) = \int_X |g|^q d\mu = \|g\|_q^q. \text{ Hence}$$

$$\begin{aligned} \|\ell_g\| &\geq \frac{|\ell_g(f)|}{\|f\|_p} = \frac{\|g\|_q^q}{\|g\|_q^{q/p}} = \|g\|_q^{q - \frac{q}{p}} \\ &= \|g\|_q^{q(1-\frac{1}{p})} = \|g\|_q. \end{aligned}$$

2) $p=1$, so $q=\infty$.

Take $0 < c < \|g\|_\infty$.

Then $\mu(\{x: |g(x)| > c\}) > 0$.

As μ is σ -finite, so $X = \bigcup_{n=1}^\infty X_n$, $\mu(X_n) < \infty$,

we can find $A = X_n \cap \{x: |g(x)| > c\}$
 s.t. $\mu(A) > 0$ and $\mu(A) < \infty$.

Consider $f = \frac{g}{|g|} \chi_A$. Then $\|f\|_1 = \mu(A)$,

$$\ell_g(f) = \int_X |g| \chi_A d\mu = \int_A |g| d\mu > c \mu(A).$$

hence $\|\ell_g\| \geq \frac{|\ell_g(f)|}{\|f\|_1} > c$. It follows

that $\|\ell_g\| \geq \|g\|_\infty$.

3) $p = \infty$, so $q = 1$.

$$\text{Define } f(x) = \begin{cases} \frac{g(x)}{|g(x)|}, & \text{if } g(x) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\|f\|_\infty = 1$ and

$$\ell_g(f) = \int_X |g| d\mu = \|g\|_1.$$

hence $\|\ell_g\| \geq \|g\|_1$. □

Therefore we can view $L^q(X, d\mu)$ as
 a subspace of $L^p(X, d\mu)^*$ using the
 isometric embedding

$$L^q(X, d\mu) \hookrightarrow L^p(X, d\mu)^*, \quad g \mapsto \ell_g.$$

Theorem

Assume $(X, \mathcal{B}, \mu)$ is a σ -finite measure
 space, $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$L^p(X, d\mu)^* = L^q(X, d\mu).$$