

Therefore we can view $L^q(X, d\mu)$ as a subspace of $L^p(X, d\mu)^*$ using the isometric embedding

$$L^q(X, d\mu) \hookrightarrow L^p(X, d\mu)^*, \quad g \mapsto \ell_g.$$

Theorem

Assume $(X, \mathcal{B}, \mu)$ is a σ -finite measure space, $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$L^p(X, d\mu)^* = L^q(X, d\mu).$$

Remark / warning

This is usually not true for $p = \infty$.

Proof of the theorem

Assume first that $\mu(X) < \infty$.

Take $\psi \in L^p(X, d\mu)$. We want to show that $\psi = f g$ for some $g \in L^q(X, d\mu)$.

As $\mu(X) < \infty$, for every $A \in \mathcal{B}$ we have

$\pi_A \in L^p(X, d\mu)$ and $\|\pi_A\|_p = \mu(A)^{1/p}$.

Define $\nu: \mathcal{B} \rightarrow \mathbb{C}$ by $\nu(A) = \psi(\pi_A)$.

This is a complex measure on $(X, \mathcal{B})$.

Indeed, clearly $\nu(\emptyset) = 0$. Assume

$A = \bigcup_{n=1}^{\infty} A_n$, $A_n \in \mathcal{B}$, $A_n \cap A_m = \emptyset$ for $n \neq m$.

Then

$$|\nu(A) - \sum_{n=1}^N \nu(A_n)| = |\psi(\pi_A) - \sum_{n=1}^N \psi(\pi_{A_n})|$$

$$= |\psi(\pi_{A \setminus \bigcup_{n=1}^N A_n})|$$

$$\leq \|\psi\| \|\pi_{A \setminus \bigcup_{n=1}^N A_n}\|_p$$

$$= \|\psi\| \mu(A \setminus \bigcup_{n=1}^N A_n)^{1/p} \xrightarrow{N \rightarrow \infty} 0.$$

Therefore $\nu(A) = \sum_{n=1}^{\infty} \nu(A_n)$.

We have $\nu \ll \mu$, since $\pi_A = 0$ in $L^p(X, d\mu)$ if $\mu(A) = 0$. By the Radon-Nikodym theorem for complex measures we get

$g \in L^1(X, d\mu)$ s.t. $\nu(A) = \int_A g d\mu$. Then

$\varphi(f) = \int_X fg d\mu$ for all simple functions f .

Consider $X_n = \{x: |g(x)| \leq n\}$.

Then $g|_{X_n} \in L^q(X_n, d\mu)$, and

$\varphi = \ell_{g|_{X_n}}$ on simple functions in $L^p(X_n, d\mu)$

As such functions are dense in $L^p(X_n, d\mu)$, we set $\varphi = \ell_{g|_{X_n}}$ on $L^p(X_n, d\mu)$. Hence

$$\|g|_{X_n}\|_q = \|\varphi|_{L^p(X_n, d\mu)}\| \leq \|\varphi\|.$$

We have $\|g|_{X_n}\|_q \xrightarrow{n \rightarrow \infty} \|g\|_q$. Indeed, if $1 < q < \infty$, this is true by the monotone convergence theorem, and if $q = \infty$ this is true by definition.

Thus $\|g\|_q \leq \|\varphi\|$, so $g \in L^q(X, d\mu)$.

As $\varphi = \ell_g$ on simple functions and they are dense in $L^p(X, d\mu)$, we set $\varphi = \ell_g$. This finishes the proof for finite measures.

Assume now that μ is σ -finite.

Thus, $X_n \uparrow X$ for some $X_n \in \mathcal{B}$, $\mu(X_n) < \infty$. Take $\varphi \in L^p(X, d\mu)^*$.

Applying the first part of the proof to $\varphi|_{L^p(X_n, d\mu)}$, we get functions $g_n \in L^p(X_n, d\mu)$ s.t. $\varphi = l_{g_n}$ on $L^p(X_n, d\mu)$. Then, if $n > m$, we have $l_{g_n}|_{X_m} = l_{g_m}$, hence $g_n|_{X_m} = g_m$ (μ -a.e.).

It follows that we can define a measurable function g s.t. $g|_{X_n} = g_n$ (μ -a.e.) for all n . Then

$$\|g\|_q = \lim_{n \rightarrow \infty} \|g|_{X_n}\|_q = \lim_{n \rightarrow \infty} \|\varphi|_{L^p(X_n, d\mu)}\| \leq \|\varphi\|,$$

so $g \in L^q(X, d\mu)$.

We also have $\varphi = l_g$ on $\bigcup_{n=1}^{\infty} L^p(X_n, d\mu)$.

But if $f \in L^p(X, d\mu)$, then ~~$f|_{X_n} \in L^p(X_n, d\mu)$~~

$f|_{X_n} \in L^p(X_n, d\mu)$ and $f|_{X_n} \rightarrow f$ in $L^p(X, d\mu)$.

Therefore $\bigcup_{n=1}^{\infty} L^p(X_n, d\mu)$ is dense in $L^p(X, d\mu)$.

Hence $\varphi = l_g$. □

Carathéodory theorem

Assume X is a set, $S \subset \mathcal{P}(X)$, $\emptyset \in S$,
and $\mu: S \rightarrow [0, +\infty]$ is a map s.t. $\mu(\emptyset) = 0$.
Can we extend μ to a measure on
 $(X, \sigma(S))$?

Define $\mu^*: \mathcal{P}(X) \rightarrow [0, +\infty]$ by

$$\mu^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \mu(A_n) : A_n \in S, A \subset \bigcup_{n=1}^{\infty} A_n \right\}$$

In general μ^* is neither a measure
nor it extends μ . But we have:

Lemma

The map μ^* has the following properties:

- 1) $\mu^*(\emptyset) = 0$;
- 2) if $A \subset B$, then $\mu^*(A) \leq \mu^*(B)$;
- 3) $\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$.

Proof

1) Taking $A_n = \emptyset$, we get $\mu^*(\emptyset) \leq \sum_{n=1}^{\infty} \mu(\emptyset) = 0$,
so $\mu^*(\emptyset) = 0$.

2) If $A \subset B$ and $B \subset \bigcup_{n=1}^{\infty} A_n$ for some $A_n \in S$,
then $A \subset \bigcup_{n=1}^{\infty} A_n$. Hence $\mu^*(A) \leq \mu^*(B)$.

3) If $\mu^*(A_n) = +\infty$ for some n , there is
nothing to prove. Therefore assume
that $\mu^*(A_n) < \infty$ for all n .

Fix $\varepsilon > 0$. Then we can find sets

$A_{nk} \in \mathcal{S}$ s.t. $A_n \subset \bigcup_{k=1}^{\infty} A_{nk}$ and

$$\sum_{k=1}^{\infty} \mu(A_{nk}) < \mu^*(A_n) + \frac{\varepsilon}{2^n}.$$

Then $\bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} A_{nk}$, so

$$\begin{aligned} \mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) &\leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu(A_{nk}) \\ &< \sum_{n=1}^{\infty} \left(\mu^*(A_n) + \frac{\varepsilon}{2^n}\right) \\ &= \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon. \end{aligned}$$

As $\varepsilon > 0$ was arbitrary, we conclude that $\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$. $\square$

Any map $\mu^*: \mathcal{P}(X) \rightarrow \{0, +\infty\}$ with properties 1)-3) is called an outer measure on X .

It turns out, every outer measure defines a measure on a certain σ -algebra.

Theorem (Carathéodory)

Assume X is a set and $\mu^*: \mathcal{P}(X) \rightarrow [0, +\infty]$ is an outer measure. Let $\mathcal{B} \subset \mathcal{P}(X)$ be the collection of sets A s.t.

$$\mu^*(B) = \mu^*(A \cap B) + \mu^*(A^c \cap B) \quad \forall B \subset X$$

(such sets A are called Carathéodory measurable with respect to μ^*).

Then $\mathcal{B}$ is a σ -algebra and $\mu \stackrel{\text{def}}{=} \mu^*|_{\mathcal{B}}$ is a measure on $(X, \mathcal{B})$.

Proof

Obviously, $\emptyset \in \mathcal{B}$, and if $A \in \mathcal{B}$, then $A^c \in \mathcal{B}$.
Let us show that $\mathcal{B}$ is also closed under finite unions (so $\mathcal{B}$ is an algebra).

Assume $A_1, A_2 \in \mathcal{B}$, and take $B \subset X$.

Note that for any $A \subset X$ we have

$$\begin{aligned}\mu^*(B) &= \mu^*((A \cap B) \cup (A^c \cap B)) \\ &\leq \mu^*(A \cap B) + \mu^*(A^c \cap B).\end{aligned}$$

In particular,

$$\mu^*(B) \leq \mu^*((A_1 \cup A_2) \cap B) + \mu^*((A_1 \cup A_2)^c \cap B).$$

On the other hand,

$$\begin{aligned}\mu^*(B) &= \mu^*(A_1 \cap B) + \mu^*(A_1^c \cap B) \\ &= \mu^*(A_1 \cap A_2 \cap B) + \mu^*(A_1 \cap A_2^c \cap B) \\ &\quad + \mu^*(A_1^c \cap A_2 \cap B) + \mu^*(A_1^c \cap A_2^c \cap B) \\ &\geq \mu^*((A_1 \cup A_2) \cap B) + \mu^*((A_1 \cup A_2)^c \cap B),\end{aligned}$$

since $(A_1 \cup A_2)^c \subset A_1^c \cap A_2^c$ and

$$A_1 \cup A_2 = (A_1 \cap A_2) \cup (A_1 \cap A_2^c) \cup (A_1^c \cap A_2).$$

hence

$$\mu^*(B) = \mu^*((A_1 \cup A_2) \cap B) + \mu^*((A_1 \cup A_2)^c \cap B),$$

so $A_1 \cup A_2 \in \mathcal{B}$.