

Finally, let us show that $\mathcal{B}$ is closed under countable unions.

Take $A_n \in \mathcal{B}$. Replacing A_n by

$$A_n \setminus \bigcup_{k=1}^{n-1} A_k = A_n \cap \left(\bigcup_{k=1}^{n-1} A_k \right)^c \in \mathcal{B},$$

we may assume that A_n are disjoint.

Let $A = \bigcup_{n=1}^{\infty} A_n$, $\tilde{A}_n = \bigcup_{k=1}^n A_k$. Take $B \subset X$.

Note that we have

$$\begin{aligned} \mu^*(\tilde{A}_n \cap B) &= \mu^*(A_n \cap \tilde{A}_n \cap B) + \mu^*(A_n^c \cap \tilde{A}_n \cap B) \\ &= \mu^*(A_n \cap B) + \mu^*(\tilde{A}_{n-1} \cap B), \end{aligned}$$

as $A_n \cap \tilde{A}_n = A_n$ and $A_n^c \cap \tilde{A}_n = \tilde{A}_{n-1}$.

Doing the same for $\mu^*(\tilde{A}_{n-1} \cap B)$ and so on, we get

$$\mu^*(\tilde{A}_n \cap B) = \sum_{k=1}^n \mu^*(A_k \cap B).$$

Then

$$\begin{aligned} \mu^*(B) &= \mu^*(\tilde{A}_n \cap B) + \mu^*(\tilde{A}_n^c \cap B) \\ &\quad (\text{as } \tilde{A}_n \in \mathcal{B}) \\ &= \sum_{k=1}^n \mu^*(A_k \cap B) + \mu^*(\tilde{A}_n^c \cap B) \\ &\geq \sum_{k=1}^n \mu^*(A_k \cap B) + \mu^*(A^c \cap B) \\ &\quad (\text{as } A^c \subset \tilde{A}_n^c) \end{aligned}$$

Letting $n \rightarrow \infty$ we get

$$\begin{aligned} \mu^*(B) &\geq \sum_{k=1}^{\infty} \mu^*(A_k \cap B) + \mu^*(A^c \cap B) \\ &\geq \mu^*(A \cap B) + \mu^*(A^c \cap B) \\ &\quad (\text{as } A \cap B = \bigcup_{k=1}^{\infty} (A_k \cap B)) \end{aligned}$$

Since the opposite inequality always holds, we conclude that $A \in \mathcal{B}$.
From the proof we also get that

$$\mu^*(B) = \sum_{k=1}^{\infty} \mu^*(A_k \cap B) + \mu^*(A^c \cap B).$$

Taking $B = A$ we obtain

$$\mu^*\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu^*(A_k).$$

Therefore $\mu^*|_{\mathcal{B}}$ is a measure. □

Def

Assume X is a set, $\mathcal{C}$ is a collection of subsets of X , $\emptyset \in \mathcal{C}$. A premeasure on $\mathcal{C}$

is a map $\mu: \mathcal{C} \rightarrow [0, +\infty]$ s.t. $\mu(\emptyset) = 0$

and $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$ whenever

A_n are mutually disjoint sets in $\mathcal{C}$ and
 $\bigcup_{n=1}^{\infty} A_n \in \mathcal{C}$.

Carathéodory extension theorem

Assume X is a set, $\mathcal{A}$ is an algebra of subsets of X , μ is a premeasure on $(X, \mathcal{A})$.
Then μ extends to a measure on $(X, \sigma(\mathcal{A}))$.

Proof

Consider the outer measure associated with μ :

$$\mu^*(A) \stackrel{\text{def}}{=} \inf \left\{ \sum_{n=1}^{\infty} \mu(A_n) : A_n \in \mathcal{A}, A \subset \bigcup_{n=1}^{\infty} A_n \right\}$$

Let $\mathcal{B}$ be the σ -algebra of Carathéodory measurable sets. To prove the theorem it suffices to show that $\mu^*|_{\mathcal{A}} = \mu$ and $\mathcal{A} \subset \mathcal{B}$, as then $\mu^*|_{\mathcal{A}}$ is the required extension.

Take $A \in \mathcal{A}$. Clearly, $\mu^*(A) \leq \mu(A)$
(take $A_1 = A, A_n = \emptyset$ for $n \geq 2$).

To prove the opposite inequality, assume $A \subset \bigcup_{n=1}^{\infty} A_n$, $A_n \in \mathcal{A}$. Define

$$\tilde{A}_n = (A \cap A_n) \setminus \bigcup_{k=1}^{n-1} A_k \in \mathcal{A}.$$

Then $\tilde{A}_n \subset A_n$, the sets $\tilde{A}_n$ are mutually disjoint and $\bigcup_{n=1}^{\infty} \tilde{A}_n = A$. Hence

$$\mu(A) = \sum_{n=1}^{\infty} \mu(\tilde{A}_n) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

It follows that $\mu(A) \leq \mu^*(A)$.

To prove that $A \subset B$ we need to show that if $A \in \mathcal{A}$, $B \subset X$, then

$$\mu^*(B) \geq \mu^*(A \cap B) + \mu^*(A^c \cap B).$$

Assume $B \subset \bigcup_{n=1}^{\infty} A_n$, $A_n \in \mathcal{A}$. Then

$$\begin{aligned} \sum_{n=1}^{\infty} \mu(A_n) &= \sum_{n=1}^{\infty} (\mu(A \cap A_n) + \mu(A^c \cap A_n)) \\ &= \sum_{n=1}^{\infty} \mu(A \cap A_n) + \sum_{n=1}^{\infty} \mu(A^c \cap A_n) \end{aligned}$$

$$\geq \mu^*(A \cap B) + \mu^*(A^c \cap B),$$

since $A \cap B \subset \bigcup_{n=1}^{\infty} A \cap A_n$ and $A^c \cap B \subset \bigcup_{n=1}^{\infty} A^c \cap A_n$.

This implies that

$$\mu^*(B) \geq \mu^*(A \cap B) + \mu^*(A^c \cap B). \quad \square$$

Remark

It can be shown that if μ is σ -finite, then $(X, \mathcal{B}, \mu^*|_{\mathcal{B}})$ coincides with the completion of $(X, \mathcal{B}(\mathcal{A}), \mu^*|_{\mathcal{B}(\mathcal{A})})$.

Is the extension of μ to $\mathcal{Z}(A)$ unique? (2)

Def

Assume X is a set. A collection $\mathcal{D}$ of subsets of X is called a Dynkin system if $\emptyset \in \mathcal{D}$, $\mathcal{D}$ is closed under complements and under countable unions of disjoint sets.

For any collection $\mathcal{S}$ of subsets of X , the intersection of all Dynkin systems containing $\mathcal{S}$ is itself a Dynkin system, called the Dynkin system generated by $\mathcal{S}$. We will denote it by $\mathcal{D}(\mathcal{S})$.

Theorem (Dynkin)

Assume X is a set and $\mathcal{S}$ is a collection of subsets of X that is closed under finite intersections. Then $\mathcal{D}(\mathcal{S})$ is a σ -algebra, and therefore $\mathcal{D}(\mathcal{S}) = \mathcal{Z}(\mathcal{S})$.

Proof

We need to show that $\mathcal{D}(\mathcal{S})$ is closed under countable unions. For this it suffices to check that $\mathcal{D}(\mathcal{S})$ is closed under finite intersections. Indeed, assume $A_n \in \mathcal{D}(\mathcal{S})$. Define $\hat{A}_n = A_n \setminus \bigcup_{k=1}^{n-1} A_k$

$= A_n \cap \bigcap_{k=1}^{n-1} A_k^c$. Assuming that $\mathcal{D}(S)$ is closed under finite intersections, we get $\tilde{A}_n \in \mathcal{D}(S)$. Then $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \tilde{A}_n \in \mathcal{D}(S)$, since the sets $\tilde{A}_n$ are mutually disjoint.

In order to show that $\mathcal{D}(S)$ is closed under finite intersections, for $A \in \mathcal{D}(S)$ define

$$\mathcal{D}_A \stackrel{\text{def}}{=} \{B \in \mathcal{D}(S) : A \cap B \in \mathcal{D}(S)\}.$$

We claim that $\mathcal{D}_A$ is a Dynkin system. Indeed, clearly $\emptyset \in \mathcal{D}_A$. If $B \in \mathcal{D}_A$, then $A \cap B^c = (A^c \cup B)^c = (A^c \cup (A \cap B))^c \in \mathcal{D}(S)$, so $B^c \in \mathcal{D}_A$. Finally, if $B_n \in \mathcal{D}_A$ are mutually disjoint, then $A \cap (\bigcup_{n=1}^{\infty} B_n) = \bigcup_{n=1}^{\infty} (A \cap B_n) \in \mathcal{D}(S)$, so $\bigcup_{n=1}^{\infty} B_n \in \mathcal{D}_A$.

Now, if $A \in S$, then $S \subset \mathcal{D}_A$, since S is closed under finite intersections. Hence $\mathcal{D}(S) \subset \mathcal{D}_A$, so $\mathcal{D}_A = \mathcal{D}(S)$. Next, if $A \in \mathcal{D}(S)$, then $S \subset \mathcal{D}_A$, since $A \in \mathcal{D}_B$ for all $B \in S$, as we already proved. Hence $\mathcal{D}(S) \subset \mathcal{D}_A$, so $\mathcal{D}_A = \mathcal{D}(S)$. This means exactly that $\mathcal{D}(S)$ is closed under finite intersections. $\square$

Corollary

(3)

Assume $(X, \mathcal{B})$ is a measurable space
and μ, ν are measures on $(X, \mathcal{B})$.

Assume $\mathcal{S} \subset \mathcal{B}$ is a collection of subsets
s.t. it is closed under finite intersections,
 $\mathcal{B} = \mathcal{B}(\mathcal{S})$, $\mu = \nu$ on $\mathcal{S}$ and there
exist $X_n \in \mathcal{S}$ s.t. $X_n \uparrow X$ and $\mu(X_n) = \nu(X_n) < \infty$
Then $\mu = \nu$ on $\mathcal{B}$.

Proof

Assume first that $X \in \mathcal{S}$ and
 $\mu(X) = \nu(X) < \infty$. Define

$$\mathcal{D} = \{A \in \mathcal{B} : \mu(A) = \nu(A)\}.$$

We claim that $\mathcal{D}$ is a Dynkin system.
Indeed, clearly $\emptyset \in \mathcal{D}$. If $A \in \mathcal{D}$, then

$$\mu(A^c) = \mu(X) - \mu(A) = \nu(X) - \nu(A) = \nu(A^c),$$

so $A^c \in \mathcal{D}$. Finally, if $A_n \in \mathcal{D}$ are mutually
disjoint, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \nu(A_n) = \nu\left(\bigcup_{n=1}^{\infty} A_n\right),$$

so $\bigcup_{n=1}^{\infty} A_n \in \mathcal{D}$.

By assumption, $\mathcal{S} \subset \mathcal{D}$, hence $\mathcal{D}(\mathcal{S}) \subset \mathcal{D}$.
By Dynkin's theorem we have $\mathcal{D}(\mathcal{S}) = \mathcal{B}(\mathcal{S}) = \mathcal{B}$.
Hence $\mathcal{D} = \mathcal{B}$, so $\mu = \nu$ on $\mathcal{B}$.

In the general case we apply the

first part of the proof to the restrictions of μ and ν to X_n and conclude that $\mu(A) = \nu(A)$ for all $A \in \mathcal{B}$ s.t. $A \subset X_n$ for some n . For arbitrary $A \in \mathcal{B}$ we then have

$$\mu(A) = \lim_{n \rightarrow \infty} \mu(A \cap X_n) = \lim_{n \rightarrow \infty} \nu(A \cap X_n) = \nu(A).$$

Combining this corollary with Carathéodory's extension theorem we get the following result.

Theorem

Assume X is a set, $\mathcal{A}$ is an algebra of subsets of X and μ is a σ -finite premeasure on $(X, \mathcal{A})$. Then μ extends uniquely to a measure on $(X, \sigma(\mathcal{A}))$.