

Riesz-Markov theorem

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Assume (X, d) is a locally compact metric space.

A linear functional $\varphi: C_c(X) \rightarrow \mathbb{C}$ is called positive if $\varphi(f) \geq 0$ for all $f \geq 0$.

Theorem (Riesz-Markov)

If $\varphi: C_c(X) \rightarrow \mathbb{C}$ is a positive linear functional, where (X, d) is a locally compact metric space, then there exists a Borel measure μ on X s.t. $\mu(K) < \infty$ for every compact $K \subset X$ and

$$\varphi(f) = \int_X f d\mu \quad \forall f \in C_c(X).$$

If X is separable, then such a measure μ is unique.

For the proof we need two auxiliary results.

Urysohn's lemma

Assume (X, d) is a metric space, $A, B \subset X$ are disjoint closed subsets. Then there exists a continuous function $f: X \rightarrow [0, 1]$ s.t. $f \equiv 1$ on A and $f \equiv 0$ on B .

Proof

Define $f(x) = \frac{d(x, B)}{d(x, A) + d(x, B)}$.

Note that this is well-defined, since

$d(x, A) + d(x, B) > 0$ for all x , as
 $\{x: d(x, A) = d(x, B) = 0\} = A \cap B = \emptyset$. $\square$

Lemma

Assume (X, d) is a compact metric space, $\mathcal{U} = (U_i)_{i=1}^n$ is a finite open cover of X (so U_i are open and $\bigcup_{i=1}^n U_i = X$). Then there exist functions $f_1, \dots, f_n$ in $C(X)$ s.t.

$0 \leq f_i \leq 1$, $\text{supp } f_i \subset U_i$, $\sum_{i=1}^n f_i(x) = 1 \forall x$.

Every such collection of functions is called a partition of unity subordinate to $\mathcal{U}$.

Proof

Let us show first that there exists an open set V_1 s.t. $\overline{V_1} \subset U_1$ and $X = V_1 \cup U_2 \cup \dots \cup U_n$.

For this, consider the closed set $K = X \setminus (U_2 \cup \dots \cup U_n) \subset U_1$. For every $x \in K$ we can find a ball $B_{r_x}(x)$ ($r_x > 0$) s.t. $\overline{B_{r_x}(x)} \subset U_1$. As K is compact, and $(B_{r_x}(x))_{x \in K}$ is an open cover of K , we can find a finite subcover $(B_{r_{x_k}}(x_k))_{k=1}^m$. Then put $V_1 = \bigcup_{k=1}^m B_{r_{x_k}}(x_k)$.

Using this construction $2n$ times ②
we can find open sets $V_1', \dots, V_n'$ and $V_1'', \dots, V_n''$ s.t.

$$\overline{V_i'} \subset V_i'', \quad \overline{V_i''} \subset U_i, \quad \bigcup_{i=1}^n V_i' = X.$$

By Urysohn's lemma, for every i ,
we can find $f_i \in C(X)$ s.t.

$$0 \leq f_i \leq 1, \quad f_i \equiv 1 \text{ on } \overline{V_i'}, \quad f_i \equiv 0 \text{ on } X \setminus V_i''.$$

Then $\text{supp } f_i \subset \overline{V_i''} \subset U_i$.

$$\text{Define } g_i(x) = \frac{f_i(x)}{\sum_{j=1}^n f_j(x)}.$$

Proof of the Riesz-Markov theorem

Assume $\varphi: C_c(X) \rightarrow \mathbb{R}$ is positive.

For open $U \subset X$, define

$$\mu(U) = \sup \{ \varphi(f) : 0 \leq f \leq 1, \text{supp } f \subset U \}.$$

For a set $A \subset X$, define then

$$\mu^*(A) = \inf_{\substack{A \subset U \\ U \text{ open}}} \mu(U).$$

Step 1 μ^* is an outer measure on X
and $\mu^*(U) = \mu(U)$ for open U .

Observe first that if $V \subset U$ are open,
then $\mu(V) \leq \mu(U)$. This implies that

$\mu^*(V) = V$ for all open V .

Obviously, $\mu^*(\emptyset) = \mu(\emptyset) = 0$.
We need to show that

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

We may assume that $\mu^*(A_n) < \infty$ for all n ,
as otherwise there is nothing to prove.

Assume first that $A_n \subseteq U_n$ are open.

Take $0 \leq f \leq 1$ s.t. $\text{supp } f \subseteq \bigcup_{n=1}^{\infty} U_n$.
($f \in C_c(X)$)

As $\text{supp } f$ is compact, there is N s.t.
 $\text{supp } f \subseteq \bigcup_{n=1}^N U_n$. Let $\rho_1, \dots, \rho_N$ be
a partition of unity in $C(\text{supp } f)$
subordinate to $(U_n \cap \text{supp } f)_{n=1}^N$. Define
 $f_n(x) = \begin{cases} \rho_n(x) f(x), & \text{if } x \in \text{supp } f, \\ 0 & \text{otherwise.} \end{cases}$

Note ^{that} the functions f_n are continuous.
(Indeed, if $x \notin \text{supp } f$ or $x \in \text{int}(\text{supp } f)$,
then $f(y) \rightarrow f(x)$ as $y \rightarrow x$, as 0
is continuous on $X \setminus \text{supp } f$ and ρ_n, f are
continuous on $\text{int}(\text{supp } f)$. Assume
now that $x \in (\text{supp } f) \setminus \text{int}(\text{supp } f)$, so
 $x \in \text{supp } f$, but $B_r(x) \setminus \text{supp } f \neq \emptyset \quad \forall r > 0$.
Then $f(x) = 0$ and $f(y) \rightarrow 0$ as $y \rightarrow x$,
since $\lim_{\substack{y \rightarrow x \\ y \in \text{supp } f}} \rho_n(y) f(y) = \rho_n(x) f(x) = 0$.)

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Note also that

$$\text{supp } f_n \subset \text{supp } f_n \subset U_n.$$

Hence $\psi(f_n) \leq \mu(U_n)$. Therefore

$$\psi(f) = \sum_{n=1}^{\infty} \psi(f_n) \leq \sum_{n=1}^{\infty} \mu(U_n) \leq \sum_{n=1}^{\infty} \mu(U_n).$$

Taking the supremum over all f s.t.

$0 \leq f \leq 1$, $\text{supp } f \subset \bigcup_{n=1}^{\infty} U_n$, we get

$$\mu\left(\bigcup_{n=1}^{\infty} U_n\right) \leq \sum_{n=1}^{\infty} \mu(U_n).$$

For general A_n , fix $\varepsilon > 0$ and choose open U_n s.t.

$$A_n \subset U_n \text{ and } \mu(U_n) < \mu^*(A_n) + \frac{\varepsilon}{2^n}.$$

Then

$$\begin{aligned} \mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) &\leq \mu^*\left(\bigcup_{n=1}^{\infty} U_n\right) \leq \sum_{n=1}^{\infty} \mu(U_n) \\ &< \sum_{n=1}^{\infty} \left(\mu^*(A_n) + \frac{\varepsilon}{2^n}\right) \\ &= \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon. \end{aligned}$$

As $\varepsilon > 0$ was arbitrary, we conclude that

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

This completes Step 1.