

Step 2 All Borel sets are Carathéodory
measurable (with respect to μ^*).
It suffices to show that every open

set U is Carathéodory measurable. We need to check that

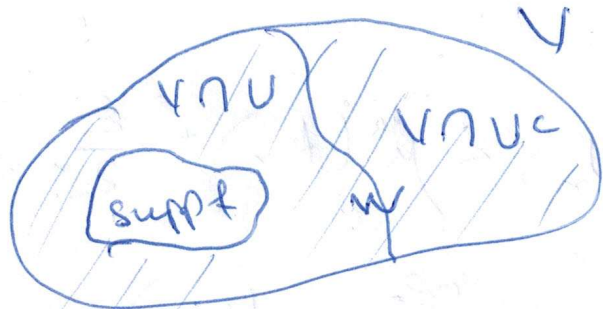
$$\mu^*(A) \geq \mu^*(A \cap U) + \mu^*(A \cap U^c) \quad \forall A \subset X.$$

Consider first an open set $A = V$.
We may assume that $\mu(V) < \infty$, as otherwise there is nothing to prove.

Fix $\varepsilon > 0$ and choose f s.t.

$$0 \leq f \leq 1, \text{ supp } f \subset V \cap U, \quad \int f \, d\mu > \mu(V \cap U) - \varepsilon$$

Consider the open set $W = V \setminus \text{supp } f$.



Choose $0 \leq g \leq 1, \text{ supp } g \subset W, \int g \, d\mu > \mu(W) - \varepsilon$.

Then

$$\begin{aligned} \mu(V \cap U) + \mu^*(V \cap U^c) &\leq \mu(V \cap U) + \mu(W) \\ &< \int f \, d\mu + \int g \, d\mu + 2\varepsilon = \int (f+g) \, d\mu + 2\varepsilon \\ &\leq \mu(V) + 2\varepsilon, \end{aligned}$$

since $0 \leq f+g \leq 1, \text{ supp } (f+g) \subset V$.

As $\varepsilon > 0$ was arbitrary, we get the required inequality.

For general A , take $V \supset A, V$ open. Then

$$\mu(V) \geq \mu(V \cap U) + \mu^*(V \cap U^c) \geq \mu^*(A \cap U) + \mu^*(A \cap U^c)$$

Taking the infimum over all V we get

$$\mu^*(A) \geq \mu^*(A \cap U) + \mu^*(A \cap U^c).$$

Step 3 If $K \subset X$ is compact, then ①
 $\mu(K) < \infty$ and $\mu(K) = \inf \{ \varphi(f) : \pi_K \leq f \}$

Also, if $0 \leq f \leq \pi_K$, then $\varphi(f) \leq \mu(K)$.

Assume $\pi_K \leq f$. Fix $\varepsilon > 0$ and consider the open set $U = \{x : f(x) > 1 - \varepsilon\}$.

Assume $0 \leq g \leq 1$, $\text{supp } g \subset U$. Then

$$g \leq \frac{f}{1-\varepsilon}, \text{ hence } \varphi(g) \leq \frac{\varphi(f)}{1-\varepsilon}.$$

Taking the supremum over all such g , we get $\mu(U) \leq \frac{\varphi(f)}{1-\varepsilon}$. Hence

$$\mu(K) \leq \mu(U) \leq \frac{\varphi(f)}{1-\varepsilon}.$$

As $\varepsilon > 0$ was arbitrary, we get $\mu(K) \leq \varphi(f)$. In particular, $\mu(K) < \infty$. (Note that $f \in C_c(X)$ s.t. $\pi_K \leq f$ exists by Urysohn's lemma.) We see also that

$$\mu(K) \leq \inf \{ \varphi(f) : \pi_K \leq f \}.$$

In order to prove the equality,

fix $\varepsilon > 0$ and choose open U s.t.

$$K \subset U, \quad \mu(K) > \mu(U) - \varepsilon.$$

By Urysohn's lemma we can find $f \in C_c(X)$ s.t. $0 \leq f \leq 1$, $f \equiv 1$ on K

(so $\pi_K \leq f$) and $\text{supp } f \subset U$. Then

$$\psi(f) \leq \mu(U) < \mu(K) + \epsilon.$$

Therefore

$$\inf \{ \psi(f) : \Pi_K \leq f \} < \mu(K) + \epsilon.$$

As $\epsilon > 0$ was arbitrary, we conclude that

$$\mu(K) = \inf \{ \psi(f) : \Pi_K \leq f \}.$$

Finally, assume $0 \leq f \leq \Pi_K$.

Then for every open $U \supset K$ we have $\text{supp } f \subset U$, so $\psi(f) \leq \mu(U)$. Hence

$$\psi(f) \leq \inf_{\substack{K \subset U \\ U \text{ open}}} \mu(U) = \mu(K).$$

Step 4 $\psi(f) = \int_X f d\mu \quad \forall f \in C_c(X).$

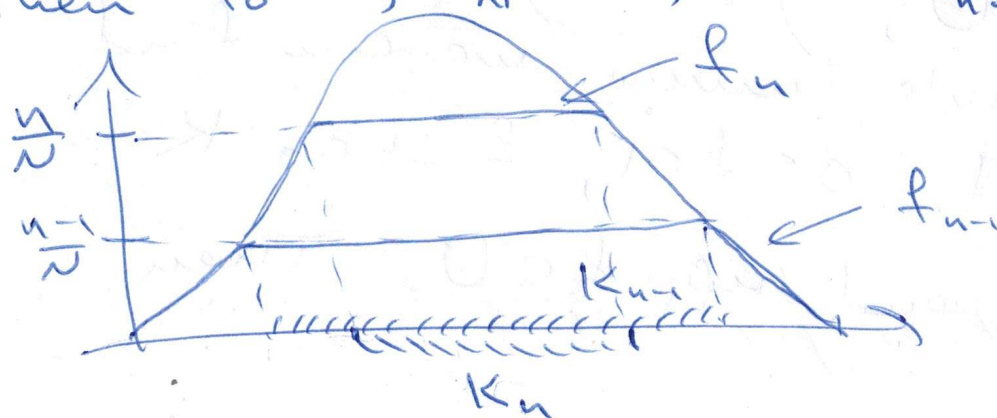
It suffices to consider $0 \leq f \leq 1$,
as such functions span $C_c(X)$.
Consider $K = \text{supp } f$ and define for some N

$$K_n = \{ x \in K : f(x) \geq \frac{n}{N} \} \quad (n=0, \dots, N),$$

$$f_n(x) = \min \{ f(x), \frac{n}{N} \} \quad (n=0, \dots, N),$$

$$g_n(x) = f_n(x) - f_{n-1}(x), \quad (n=1, \dots, N).$$

Then $f_0 = 0$, $f_N = f$, so $f = \sum_{n=1}^N g_n$.



we have (2)

$$\frac{1}{N} \pi_{K_n} \leq g_n \leq \frac{1}{N} \pi_{K_{n-1}} \quad (n=1, \dots, N)$$

By integrating we get

$$\frac{1}{N} \mu(K_n) \leq \int_X g_n d\mu \leq \frac{1}{N} \mu(K_{n-1}).$$

On the other hand by step 3 we have

$$\frac{1}{N} \mu(K_n) \leq \psi(g_n) \leq \frac{1}{N} \mu(K_{n-1}).$$

It follows that both $\sum_X f d\mu$ and $\psi(f)$ lie in the interval

$$\left[\frac{1}{N} \sum_{n=1}^N \mu(K_n), \frac{1}{N} \sum_{n=1}^N \mu(K_{n-1}) \right].$$

The length ~~of~~ of this interval is

$$\frac{1}{N} (\mu(K_0) - \mu(K_N)) \leq \frac{1}{N} \mu(K_0) = \frac{\mu(K)}{N}.$$

Therefore

$$\left| \sum_X f d\mu - \psi(f) \right| \leq \frac{\mu(K)}{N}.$$

Letting $N \rightarrow \infty$ we conclude that

$$\psi(f) = \sum_X f d\mu.$$

Step 5 If X is separable, then μ is uniquely determined by ψ .

Assume μ is a Borel measure s.t.

$\mu(K) < \infty$ $\forall$ compact $K \subset X$ and

$$\psi(f) = \sum_X f d\mu \quad \forall f \in C_c(X).$$

Recall from the lecture on regularity of measures that then μ is regular (as X is separable, locally compact).

In particular,

$$\mu(A) = \inf_{\substack{A \subset U \\ U \text{ open}}} \mu(U) \quad \forall A \in \mathcal{B}(X).$$

Therefore it suffices to show that $\mu(U)$ is determined by φ .

If $\overline{U}$ is compact, then $f_n \uparrow \chi_U$,

where $f_n(x) = \frac{\text{nd}(x, U^c)}{1 + \text{nd}(x, U^c)}$. Hence

$$\mu(U) = \lim_{n \rightarrow \infty} \int_X f_n d\mu = \lim_{n \rightarrow \infty} \varphi(f_n).$$

For general U , we can find open U_n

s.t. $\overline{U_n}$ are compact and $U_n \uparrow U$

(again, recall the lecture on regularity).

Then $\mu(U_n) \uparrow \mu(U)$, so $\mu(U)$ is determined by φ . □

Remark

Without separability the uniqueness is not always true. It can be checked that the measure we constructed is the most has the following properties:

- 1) $\mu(K) < \infty$ if compact $K \subset X$;

2) μ is outer regular ($\mu(A) = \inf_{\substack{A \subset U \\ U \text{ open}}} \mu(U)$) ; ⁽³⁾

3) μ is inner regular on open sets ;

$$\mu(U) = \sup_{\substack{K \subset U \\ K \text{ compact}}} \mu(K) \quad \forall \text{ open } U$$

K compact

(This is where we need the full strength of Step 3)

Such measures are called Radon measures

It can be shown that the uniqueness holds within the ~~class~~ class of Radon measures.