

Dual of $C(X)$

As an application of the Riesz-Markov theorem we will describe $C(X)^*$ in terms of measures for compact metric spaces (X, d) .

Denote by $M(X)$ the space of complex Borel measures on X .

For every $\nu \in M(X)$ we want to make sense of $\int_X f d\nu$ for $f \in C(X)$.

It is enough to consider finite signed measures, as then we can define

$$\int_X f d\nu = \int_X f d(\operatorname{Re} \nu) + i \int_X f d(\operatorname{Im} \nu).$$

So assume ν is a finite signed measure.

Then $\nu = \mu_1 - \mu_2$ for positive measures

and we define

$$\int_X f d\nu = \int_X f d\mu_1 - \int_X f d\mu_2.$$

This is well-defined, since if

$$\nu = \mu_1 - \mu_2 = \omega_1 - \omega_2,$$

then $\mu_1 + \omega_2 = \mu_2 + \omega_1$ and

$$\int_X f d\mu_1 + \int_X f d\omega_2 = \int_X f d\mu_2 + \int_X f d\omega_1.$$

Thus, every $\nu \in M(X)$ defines a linear functional

$$\ell_\nu : C(X) \rightarrow \mathbb{C} \quad \text{by } \ell_\nu(f) = \int_X f d\nu,$$

and the map ~~map~~ $\nu \mapsto \ell_\nu$ is linear.

Lemma

If $\nu \in M(X)$ and $d\nu = g d|\nu|$ is its polar decomposition, then

$$\int_X f d\nu = \int_X f g d|\nu| \quad \forall f \in C(X).$$

Proof

More generally, let us show that if μ is a finite Borel measure, $g \in L^1(X, d\mu)$, and $d\nu = g d\mu$, then

$$\int_X f d\nu = \int_X f g d\mu \quad \forall f \in C(X).$$

It suffices to consider $g \geq 0$.

Then ν is positive, so $\int_X f d\nu$ has the usual meaning.

~~Lemma~~ We will then prove an even more general statement! (9)

$$\int_X f d\nu = \int_X f g d\mu \quad \forall f \in L^\infty(X, \mu).$$

As simple functions are dense in $L^\infty(X, \mu)$, it suffices to consider ~~simple~~ simple f , hence it suffices to consider $f = \mathbb{1}_A$. Then

$$\int_X \mathbb{1}_A d\nu = \nu(A) = \int_X \mathbb{1}_A g d\mu = \int_X \mathbb{1}_A g d\mu$$

by the definition
of $d\nu = g d\mu$



Lemma

For every $\nu \in M(X)$, the linear functional φ_ν is bounded and $\|\varphi_\nu\| = |\nu|(X)$.

Recall that the norm on $C(X)$ is $\|f\| = \sup_{x \in X} |f(x)|$.

Proof

we have, with $d\nu = g d|\nu|$ being the polar decomposition,

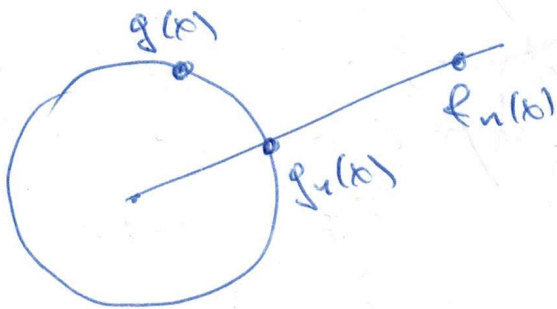
$$\begin{aligned} |\varphi_\nu(f)| &= \left| \int_X f d\nu \right| = \left| \int_X f g d|\nu| \right| \\ &\leq \int_X |f| d|\nu| \leq \|f\| \int_X 1 d|\nu| \\ &\leq \|f\| |\nu|(X). \end{aligned}$$

Therefore $\|\varphi_\nu\| \leq |\nu|(X)$.

For the opposite inequality, recall that $|v|$ is regular and $C(X)$ is dense in $L^1(X, d|v|)$. Hence we can find $f_n \in C(X)$ s.t. $f_n \rightarrow g$ in $L^1(X, d|v|)$. Define

$$g_n(x) = \begin{cases} f_n(x), & \text{if } |f_n(x)| \leq 1, \\ \frac{f_n(x)}{|f_n(x)|}, & \text{if } |f_n(x)| > 1. \end{cases}$$

Then $\|g_n\| \leq 1$ and $\|g - g_n\| \leq \|g - f_n\|$.



Therefore $\|g - g_n\|_1 \leq \|g - f_n\|_1 \xrightarrow{n \rightarrow \infty} 0$.

We have

$$\varphi_v(\bar{g}_n) = \sum_X \bar{g}_n g d|v| \rightarrow \sum_X |d|v| = |v|(X),$$

since

$$\left| \sum_X (\bar{g}_n g - 1) d|v| \right| \leq \sum_X |\bar{g}_n g - 1| d|v|$$

$$= \sum_X |\bar{g}_n - \bar{g}| d|v| = \|g_n - g\|_1 \xrightarrow{n \rightarrow \infty} 0$$

as $|g(x)| = 1$

Therefore $\|\varphi_v\| \geq \sup_n |\varphi_v(\bar{g}_n)| \geq |v|(X)$.

Lemma

Assume X is a compact metric space.

1) If $\varphi: C(X) \rightarrow \mathbb{C}$ is a positive linear functional, then φ is bounded and $\|\varphi\| = \varphi(1)$.

2) Every bounded linear functional on $C(X)$ is a linear combination of positive linear functionals.

Proof

1) Define $(\cdot, \cdot): C(X) \times C(X) \rightarrow \mathbb{C}$ by
 $(f, g) = \varphi(f \bar{g})$.

This is a pre-inner product on $C(X)$ by positivity of φ . Hence, by the Cauchy-Schwarz inequality we have

$$|(f, g)| \leq (f, f)^{1/2} (g, g)^{1/2}.$$

Applying this to $g = 1$ we get

$$|\varphi(f)| \leq \varphi(|f|^2)^{1/2} \varphi(1)^{1/2}$$

As $|f|^2 \leq \|f\|^2 \cdot 1$, we have

$$\varphi(|f|^2) \leq \|f\|^2 \varphi(1).$$

Therefore

$$|\varphi(f)| \leq \|f\| \varphi(1).$$

This shows that φ is bounded and

$$\| \varphi \| \leq \varphi(1).$$

On the other hand, $\| \varphi \| \geq \varphi(1)$, since $\| 1 \| = 1$. Hence $\| \varphi \| = \varphi(1)$.

2) For $\varphi \in C(X)^*$ define linear functionals φ_1, φ_2 by

$$\varphi_1(f) = \frac{1}{2} (\varphi(f) + \overline{\varphi(\bar{f})}),$$

$$\varphi_2(f) = \frac{1}{2i} (\varphi(f) - \overline{\varphi(\bar{f})}).$$

Then $\varphi_1, \varphi_2 \in C(X)^*$, $\varphi = \varphi_1 + i\varphi_2$ and

$$\varphi_j(\bar{f}) = \overline{\varphi_j(f)} \text{ for } f \in C(X) \text{ and } j=1,2.$$

Therefore it suffices to show that if

$\varphi \in C(X)^*$ satisfies $\varphi(\bar{f}) = \overline{\varphi(f)}$, then

φ is a linear combination of positive linear functionals. This is shown by

an analogue of Jordan's decomposition as follows:

Define, for $f \in C(X)$, $f \geq 0$,

$$\varphi_+(f) \stackrel{\text{def}}{=} \sup \{ \varphi(g) : 0 \leq g \leq f \}.$$

Note that $\varphi_+(f)$ is finite, since if

$0 \leq g \leq f$, then $|\varphi(g)| \leq \| \varphi \| \| g \| \leq \| \varphi \| \| f \|$.

Since we can take $g \geq 0$, we have $\varphi_+(f) \geq 0$.

Obviously, $\varphi_+(t f) = t \varphi_+(f)$ ②

for all $f \in C(X)$, $f \geq 0$, and $t \geq 0$.

Let us show next that if $f_1, f_2 \geq 0$, then

$$\varphi_+(f_1 + f_2) = \varphi_+(f_1) + \varphi_+(f_2).$$

If $0 \leq g_1 \leq f_1$, $0 \leq g_2 \leq f_2$, then $0 \leq g_1 + g_2 \leq f_1 + f_2$.

This implies that

$$\varphi_+(f_1 + f_2) \geq \varphi_+(f_1) + \varphi_+(f_2).$$

To prove the opposite inequality, take $0 \leq g \leq f_1 + f_2$. Define

$$g_1 = \min\{g, f_1\} \quad \text{and} \quad g_2 = g - g_1 \geq 0$$

We have $0 \leq g_1 \leq f_1$. Also, for every $x \in X$,

$$g_2(x) = g(x) - g_1(x) = \begin{cases} 0, & \text{if } g(x) \leq f_1(x) \\ g(x) - f_1(x), & \text{if } g(x) > f_1(x) \end{cases}$$

$$\leq \underbrace{(f_1(x) + f_2(x)) - f_1(x)}_{= f_2(x)} = f_2(x).$$

Then

$$\varphi(g) = \varphi(g_1) + \varphi(g_2) \leq \varphi_+(f_1) + \varphi_+(f_2).$$

$$\text{Hence } \varphi_+(f_1 + f_2) \leq \varphi_+(f_1) + \varphi_+(f_2).$$

This proves that $\varphi_+(f_1 + f_2) = \varphi_+(f_1) + \varphi_+(f_2)$.

We can now extend φ_+ to a positive linear functional on $C(X)$ in the same way, as we extended integrals from positive

functions:

First we extend φ_+ to real-valued functions by

$$\varphi_+(f_1 - f_2) \stackrel{\text{def}}{=} \varphi_+(f_1) - \varphi_+(f_2)$$

for $f_1, f_2 \geq 0$. This is well-defined, since

$$\text{if } f_1 - f_2 = g_1 - g_2 \quad (f_1, f_2, g_1, g_2 \geq 0),$$

then $f_1 + g_2 = f_2 + g_1$ and hence

$$\varphi_+(f_1) + \varphi_+(g_2) = \varphi_+(f_2) + \varphi_+(g_1).$$

Then we extend φ_+ to complex-valued functions by

$$\varphi_+(f) \stackrel{\text{def}}{=} \varphi_+(\operatorname{Re} f) + i\varphi_+(\operatorname{Im} f).$$

Finally, consider $\varphi_- \stackrel{\text{def}}{=} \varphi_+ - \varphi$.

For $f \geq 0$ we have, by definition,

$$\varphi_+(f) \geq \varphi(f), \text{ hence } \varphi_- \text{ is positive.}$$

We also have $\varphi = \varphi_+ - \varphi_-$. □

Theorem

Assume X is a compact metric space.

Consider the space $M(X)$ of complex Borel

measures on X . Then the map

$$M(X) \rightarrow C(X)^*, \quad \nu \mapsto \varphi_\nu, \quad \varphi_\nu(f) \stackrel{\text{def}}{=} \int_X f d\nu,$$

is a linear isomorphism.

Proof

③

We have already shown that $\|\varphi_v\| = \|v\|_1(X)$.
In particular, $\varphi_v = 0$ if and only if $v = 0$.

This shows that the map

$$M(X) \rightarrow C(X)^*, v \mapsto \varphi_v,$$

is injective. The image of this map contains all positive linear functionals by the Riesz-Markov theorem. Since such functionals span the entire space $C(X)^*$ by the previous lemma, it follows that the ~~image~~ map $M(X) \rightarrow C(X)^*$ is surjective. $\square$

Corollary

The vector space $M(X)$ is a Banach space with norm $\|v\| = \|v\|_1(X)$.

Remark

More generally, similar arguments show that we have a linear isomorphism $M_{\text{reg}}(X) \cong C_0(X)^*$ for every locally compact metric space X , where $M_{\text{reg}}(X)$ denotes the space of regular complex Borel measures on X , where v is called regular when $|v|$ is regular (equivalently, $(\operatorname{Re} v)_\pm, (\operatorname{Im} v)_\pm$ are regular).