

The claim implies that the left and right derivatives φ'_- and φ'_+ exist and

$$\frac{\varphi(t_1) - \varphi(x)}{t_1 - x} \leq \varphi'_-(x) \leq \varphi'_+(x) \leq \frac{\varphi(t_2) - \varphi(x)}{t_2 - x}$$

for $t_1 < x < t_2$. From the existence of φ'_- , φ'_+

we conclude that φ is continuous.

We see also that for $l(s) = \varphi'_+(x)(s-x) + \varphi(x)$

we have $\varphi \geq l$ and $\varphi(x) = l(x)$. $\square$

Jensen's inequality

Assume $(X, \mathcal{B}, \mu)$ is a probability measure space (so $\mu(X) = 1$), $\varphi: [0, +\infty) \rightarrow [0, +\infty)$ is a convex function. Then, for every integrable function $f: X \rightarrow [0, +\infty)$ we have

$$\varphi\left(\int_X f d\mu\right) \leq \int_X \varphi \circ f d\mu.$$

The same inequality holds for any measurable $f: X \rightarrow [0, +\infty]$ if

$\lim_{x \rightarrow +\infty} \varphi(x) = +\infty$ and we put $\varphi(+\infty) = +\infty$.

Proof

Note that as φ is continuous on $(0, +\infty)$, the function $\varphi \circ f$ is measurable.

If $\int_X f d\mu = 0$, so $f = 0$ a.e., then there is nothing to prove, as then

$$\varphi\left(\int_X f d\mu\right) = \varphi(0), \quad \int_X \varphi \circ f d\mu = \varphi(0) \mu(X) = \varphi(0).$$

Assume therefore that $f: X \rightarrow (0, +\infty)$

and $t = \int_X f d\mu$ satisfies $0 < t < \infty$.

Take a linear function $l(s) = \alpha s + \beta$

s.t. $\varphi \geq l$ on $(0, +\infty)$. Then also $\varphi(0) \geq l(0) = \beta$

(Indeed, for $s > 0$ we have

$$l(s) \leq \varphi(s) \leq (1-s)\varphi(0) + s\varphi(1),$$

and letting $s \downarrow 0$ we get $\varphi(0) \geq l(0)$.)

Then

$$\begin{aligned} l\left(\int_X f d\mu\right) &= \alpha \int_X f d\mu + \beta \\ &= \int_X l \circ f d\mu \leq \int_X \varphi \circ f d\mu. \end{aligned}$$

Taking the supremum over l as above, by the previous lemma we get

$$\varphi\left(\int_X f d\mu\right) \leq \int_X \varphi \circ f d\mu.$$

Assume now that $\lim_{x \rightarrow +\infty} \varphi(x) = +\infty$

and put $\varphi(+\infty) = +\infty$.

If $f: X \rightarrow [0, +\infty]$ and $\int_X f d\mu < \infty$
(so $f < +\infty$ a.e.), then the same proof
as above gives the required inequality.

It remains to show that if

$$\int_X f d\mu = +\infty, \text{ then also } \int_X \varphi \circ f d\mu = +\infty.$$

For this it suffices to show that there
is $\ell(s) = \alpha s + \beta$ s.t. $\varphi \geq \ell$ and $\alpha > 0$,
as then

$$\int_X \varphi \circ f d\mu \geq \alpha \int_X f d\mu + \beta = +\infty.$$

Such an ℓ exists by the proof of
the previous lemma: take $t > 0$
and find x s.t. $\varphi(x) > \varphi(t)$. Then

$$\varphi'_+(x) \geq \frac{\varphi(t) - \varphi(x)}{t - x} > 0 \quad \text{and}$$

we can take $\ell(s) = \varphi'_+(x)(s - x) + \varphi(x)$. □

①

Lemma

Assume $f \in L^1(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$ ($1 \leq p \leq \infty$).
 Then the function $y \mapsto f(y)g(x-y)$ is
 integrable for a.e. x , $f * g \in L^p(\mathbb{R}^n)$ and
 $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.

Proof

For $p = \infty$ the lemma is easy, since for every $x \in \mathbb{R}^n$,
 $|f(y)g(x-y)| \leq |f(y)| \|g\|_\infty$ for a.e. y .

Assume $1 < p < \infty$ (for $p = 1$ we already know
 the lemma).

For everything except the last inequality
 we may assume that $f, g \geq 0$.

If $f = 0$ a.e., there is nothing to prove,
 so we may assume that $\|f\|_1 > 0$ and
 then replacing f by $\frac{f}{\|f\|_1}$, that $\|f\|_1 = 1$.

By Jensen's inequality,

$$\left(\int_{\mathbb{R}^n} f(y)g(x-y) dy \right)^p \leq \int_{\mathbb{R}^n} f(y)g(x-y)^p dy \\ = (f * g^p)(x).$$

As $g^p \in L^1(\mathbb{R}^n)$, it follows that

$$\int_{\mathbb{R}^n} f(y)g(x-y) dy < \infty \text{ for a.e. } x \text{ and} \\ \|f * g\|_p \leq \left(\int_{\mathbb{R}^n} (f * g^p)(x) dx \right)^{1/p} \\ = \left(\|f\|_1 \|g^p\|_1 \right)^{1/p} = \|g^p\|_1^{1/p} = \|g\|_p.$$

Finally, for general $f \in L^1(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$ we have

$$|f * g| \leq |f| * |g|,$$

$$\text{so } \|f * g\|_p \leq \| |f| * |g| \|_p \leq \| |f| \|_1 \cdot \| |g| \|_p = \|f\|_1 \cdot \|g\|_p. \quad \square$$

Note that ~~that~~

$$\int_{\mathbb{R}^n} f(y) g(x-y) dy = \int_{\mathbb{R}^n} f(x-y) g(y) dy,$$

$$\text{so } f * g = g * f.$$

Remark

More generally, for $\mu \in M(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, we can define $\mu * g = g * \mu \in L^p(\mathbb{R}^n)$ by

$$(\mu * g)(x) = \int_{\mathbb{R}^n} g(x-y) d\mu(y).$$

$$\text{Then } \|\mu * g\|_p \leq |\mu|(\mathbb{R}^n) \|g\|_p.$$

Proposition

If $f, g \in L^1(\mathbb{R}^n)$, then $\widehat{f * g} = (2\pi)^n \widehat{f} \widehat{g}$.

Proof

$$\widehat{f * g}(x) = \int_{\mathbb{R}^n} (f * g)(y) e^{-i\langle x, y \rangle} dy$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(z) g(y-z) e^{-i\langle x, y \rangle} dz \right) dy$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(z) e^{-i\langle x, z \rangle} g(y-z) e^{-i\langle x, y-z \rangle} dy \right) dz$$

$$= (2\pi)^n \widehat{f} \widehat{g}. \quad \square$$

What are convolutions good for?

(2)

Example

Consider $f = \frac{1}{\lambda_n(B_r(0))} \mathbb{1}_{B_r(0)}$. Then

$$\begin{aligned} (f * g)(x) &= \frac{1}{\lambda_n(B_r(0))} \int_{B_r(0)} g(x-y) dy \\ &= \frac{1}{\lambda_n(B_r(x))} \int_{B_r(x)} g(y) dy. \end{aligned}$$

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$,
write ∂^α for ~~$\frac{\partial^{\alpha_1 + \dots + \alpha_n}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$~~ $\frac{\partial^{\alpha_1 + \dots + \alpha_n}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$.

Denote by $L^1_{loc}(\mathbb{R}^n)$ the space of Lebesgue measurable functions that are integrable on every ball. We identify functions that coincide a.e. (so, more pedantically, $L^1_{loc}(\mathbb{R}^n)$ is a space of equivalence classes of functions). We have $L^p(\mathbb{R}^n) \subset L^1_{loc}(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$.

Lemma
If $\phi \in C_c^\infty(\mathbb{R}^n)$ and $f \in L^1_{loc}(\mathbb{R}^n)$, then $\phi * f \in C^\infty(\mathbb{R}^n)$ and $\partial^\alpha(\phi * f) = (\partial^\alpha \phi) * f$.

Proof

It suffices to show that $\frac{\partial}{\partial x_i}(\phi * f) = \frac{\partial \phi}{\partial x_i} * f$.

Let $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^n$.

Then

$$\frac{\partial (\phi * f)}{\partial x_i}(x) = \frac{d}{dt} (\phi * f)(x + te_i)$$

$$= \frac{d}{dt} \int_{\mathbb{R}^n} \phi(x + te_i - y) f(y) dy$$

$$= \int_{\mathbb{R}^n} \frac{\partial \phi}{\partial x_i}(x - y) f(y) dy = \left(\frac{\partial \phi}{\partial x_i} * f \right)(x),$$

where we used the theorem on differentiation of integrals depending on parameters. $\Rightarrow$

By choosing suitable ϕ we can make $\phi * f$ close to f , as we will see shortly.

Def

A positive mollifier is a function $\phi \in C_c(\mathbb{R}^n)$

s.t. $\phi \geq 0$ and $\int_{\mathbb{R}^n} \phi(x) dx = 1$.

For a function ϕ on $\mathbb{R}^n$ and $\varepsilon > 0$, define

$$\phi^\varepsilon(x) = \varepsilon^{-n} \phi\left(\frac{x}{\varepsilon}\right).$$

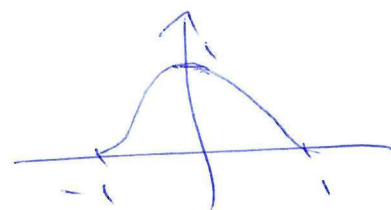
Note that if $\phi \in L^1(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \phi dx = 1$, then

$$\int_{\mathbb{R}^n} \phi^\varepsilon dx = 1.$$

Example

Consider the function h on $\mathbb{R}$ defined by

$$h(t) = \begin{cases} e^{-\frac{1}{1-t^2}}, & |t| < 1, \\ 0, & |t| \geq 1. \end{cases}$$



Then $h \in C_c^\infty(\mathbb{R})$. Hence $\phi(x) = c_n h(x)$ is a mollifier, where $c_n = \left(\int_{\mathbb{R}^n} h(|x|) dx \right)^{-1}$.