

Proposition

Let $\phi \in L^1(\mathbb{R}^n)$ be s.t. $\phi \geq 0$ and $\int_{\mathbb{R}^n} \phi dx = 1$. ③

Then we have:

1) if $f \in C_0(\mathbb{R}^n)$, then $\phi^\varepsilon * f \in C_0(\mathbb{R}^n)$ and

$$\|\phi^\varepsilon * f - f\| \xrightarrow{\varepsilon \downarrow 0} 0 \quad (\text{uniform norm});$$

2) if $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, then

$$\|\phi^\varepsilon * f - f\|_p \xrightarrow{\varepsilon \downarrow 0} 0.$$

Proof

Since $\|\phi^\varepsilon * f\| \leq \|f\|$, $\|\phi^\varepsilon * f\|_p \leq \|f\|_p$ and $C_c(\mathbb{R}^n)$ is dense in $C_0(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ ($1 \leq p < \infty$), it suffices to prove the proposition for $f \in C_c(\mathbb{R}^n)$. Let $\varepsilon > 0$ be s.t. $\text{supp } f \subset B_R(0)$.

Assume first that ϕ is zero outside $B_R(0)$ for some $R > 0$. Then ϕ^ε is zero outside $B_{\varepsilon R}(0)$.

As f is uniformly continuous, for every $\delta > 0$ we can find $\varepsilon_0 > 0$ s.t. if $|x - y| < \varepsilon_0 R$, then

$$|f(x) - f(y)| < \delta.$$

Then, if $|x - y| < \varepsilon_0 R$, we have for ~~all~~ $\varepsilon > 0$ that

$$|(\phi^\varepsilon * f)(x) - (\phi^\varepsilon * f)(y)| = \left| \int_{\mathbb{R}^n} (f(x-z) - f(y-z)) \phi^\varepsilon(z) dz \right|$$

$$\leq \int_{\mathbb{R}^n} |f(x-z) - f(y-z)| \phi^\varepsilon(z) dz$$

$$\leq \delta \int_{\mathbb{R}^n} \phi^\varepsilon(z) dz = \delta.$$

This implies that $\phi^\varepsilon * f$ is continuous.

Similarly, for $x \in \mathbb{R}^n$ and $\varepsilon < \varepsilon_0$,

$$\begin{aligned}
|(\varphi^\varepsilon * f)(x) - f(x)| &= \left| \int_{\mathbb{R}^n} f(x-y) \varphi^\varepsilon(y) dy - f(x) \int_{\mathbb{R}^n} \varphi^\varepsilon(y) dy \right| \\
&= \left| \int_{\mathbb{R}^n} f(x-y) \varphi^\varepsilon(y) dy - f(x) \int_{\mathbb{R}^n} \varphi^\varepsilon(y) dy \right| \\
&\leq \int_{\mathbb{R}^n} |f(x-y) - f(x)| \varphi^\varepsilon(y) dy \\
&\leq \int_{\mathbb{R}^n} \varphi^\varepsilon(y) dy = 1, \text{ as } \varphi^\varepsilon \text{ is zero outside } B_{\varepsilon_0} \mathbb{R}
\end{aligned}$$

Therefore $\varphi^\varepsilon * f \xrightarrow{\varepsilon \downarrow 0} f$ uniformly on $\mathbb{R}^n$.

As $\text{supp}(\varphi^\varepsilon * f) \subset B_{\varepsilon R + r}(0)$, it follows also that $\varphi^\varepsilon * f \xrightarrow{\varepsilon \downarrow 0} f$ in $L^p(\mathbb{R}^n)$ for all $1 \leq p < \infty$.

For general φ , fix $\delta > 0$ and choose $R > 0$ s.t. $\int_{\mathbb{R}^n \setminus B_R(0)} \varphi dx < \delta$.

Write $\varphi = \varphi_1 + c \varphi_2$, where $\varphi_1 = \varphi \chi_{\mathbb{R}^n \setminus B_R(0)}$, $c = \int_{B_R^*(0)} \varphi dx$ (so $1 - \delta \leq c \leq 1$)

and $\varphi_2 = \frac{1}{c} \varphi \chi_{B_R(0)}$. Then we know already that $\varphi_2^\varepsilon * f \rightarrow f$ in $C_0(\mathbb{R}^n)$

and $L^p(\mathbb{R}^n)$ ($1 \leq p < \infty$). Hence

$$\begin{aligned}
\|\varphi^\varepsilon * f - f\| &\leq \|\varphi_1^\varepsilon * f\| + c \|\varphi_2^\varepsilon * f - f\| + (1-c) \|f\| \\
&\leq \delta \|f\| + c \|\varphi_2^\varepsilon * f - f\| + \delta \|f\|.
\end{aligned}$$

As $\delta > 0$ was arbitrary, it follows that

$\varphi^\varepsilon * f - f \xrightarrow{\varepsilon \downarrow 0} 0$ in $C_0(\mathbb{R}^n)$. The

same proof with $\|\cdot\|$ replaced by $\|\cdot\|_p$,
shows that $\varphi^\varepsilon * f - f \xrightarrow{\varepsilon \downarrow 0} 0$ in $L^p(\mathbb{R}^n)$,
($1 \leq p < \infty$). $\square$

Corollary

For any Radon measure μ on $\mathbb{R}^n$,
 $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, d\mu)$ for $1 \leq p < \infty$.

Proof

We already know that $C_c(\mathbb{R}^n)$ is
dense in $L^p(\mathbb{R}^n, d\mu)$. On the other
hand, if $f \in C_c(\mathbb{R}^n)$, $\text{supp } f \subset B_r(0)$,
 $\varphi \in C_c^\infty(\mathbb{R}^n)$ is a positive mollifier,
 $\text{supp } \varphi \subset B_R(0)$, then $\varphi^\varepsilon * f \in C_c^\infty(\mathbb{R}^n)$,
 $\varphi^\varepsilon * f \rightarrow f$ uniformly on $\mathbb{R}^n$ and
 $\text{supp } (\varphi^\varepsilon * f) \subset B_{r+R}(0)$ if $\varepsilon \leq 1$,
hence $\varphi^\varepsilon * f \rightarrow f$ in $L^p(\mathbb{R}^n, d\mu)$. $\square$

Corollary (Riemann-Lebesgue lemma)

①

If $f \in L^1(\mathbb{R}^n)$, then $\hat{f} \in C_0(\mathbb{R}^n)$.

Proof

We already know that $\hat{f} \in C_b(\mathbb{R}^n)$ (the space of bounded continuous functions) and $\|\hat{f}\|_\infty \leq \|f\|_1$.

Consider $\varphi(x) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{|x|^2}{2}}$, so $\int_{\mathbb{R}^n} \varphi(x) dx = 1$.

We know that $\hat{\varphi} = \frac{1}{(2\pi)^{n/2}} \varphi \in C_0(\mathbb{R}^n)$.

Then $\widehat{\varphi^\varepsilon \times f}(x) = \hat{\varphi}(\varepsilon x) = \frac{1}{(2\pi)^{n/2}} \varphi(\varepsilon x)$, $\hat{\varphi}^\varepsilon \in C_0(\mathbb{R}^n)$.

As $\varphi^\varepsilon \times f \xrightarrow{\varepsilon \downarrow 0} f$ in $L^1(\mathbb{R}^n)$, we have

$$\|\widehat{\varphi^\varepsilon \times f} - \hat{f}\|_\infty \xrightarrow{\varepsilon \downarrow 0} 0.$$

But $\widehat{\varphi^\varepsilon \times f} = (2\pi)^n \hat{\varphi}^\varepsilon \hat{f} \in C_0(\mathbb{R}^n)$,

hence $\hat{f} \in C_0(\mathbb{R}^n)$. $\square$

Remark

Another possibility is to approximate f by linear combinations of $\chi_{[a,b] \times \dots \times [a_n, b_n]}$.

Note that for $\chi_{[a,b]} \in L^1(\mathbb{R})$ we have

$$\widehat{\chi_{[a,b]}}(x) = \frac{1}{2\pi} \int_a^b e^{-ixy} dy = \frac{e^{-iax} - e^{-ibx}}{2\pi i x} \xrightarrow{x \rightarrow \infty} 0.$$

Fourier inversion theorem

Assume $f \in L^1(\mathbb{R}^n)$ is s.t. $\hat{f} \in L^1(\mathbb{R}^n)$.

Then, for a.e. x , we have

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(y) e^{i\langle x, y \rangle} dy.$$

Equivalently,

$$\hat{\hat{f}}(x) = \frac{1}{(2\pi)^n} f(-x).$$

Lemma

For any $f, g \in L^1(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} \hat{f} g dx = \int_{\mathbb{R}^n} f \hat{g} dx.$$

(Note that $\hat{f} g \in L^1(\mathbb{R}^n)$, as $\hat{f}$ is bounded.)

Proof

This follows by Fubini's theorem!

$$\begin{aligned} \int_{\mathbb{R}^n} \hat{f}(x) g(x) dx &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(y) g(x) e^{-i\langle x, y \rangle} dy \right) dx \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(y) g(x) e^{-i\langle x, y \rangle} dx \right) dy \\ &= \int_{\mathbb{R}^n} f(y) \hat{g}(y) dy. \quad \square \end{aligned}$$

Proof of the theorem

We already know that the theorem is true for $\phi(x) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{|x|^2}{2}}$, hence

also for $\phi \in \mathcal{S}$ (so $\phi^\varepsilon(x) = \varepsilon^{-n} \phi(x/\varepsilon)$)

Consider the function I_ε defined by ②

$$I_\varepsilon(x) = \int_{\mathbb{R}^n} \widehat{\varphi^\varepsilon * f}(y) e^{i\langle x, y \rangle} dy \\ = (2\pi)^n \int_{\mathbb{R}^n} \widehat{\varphi^\varepsilon}(y) \widehat{f}(y) e^{i\langle x, y \rangle} dy$$

On the one hand,

$$(2\pi)^n \widehat{\varphi^\varepsilon}(y) = (2\pi)^n \widehat{\varphi}(\varepsilon y) = e^{-\frac{\varepsilon^2 |y|^2}{2}}.$$

As $\widehat{f} \in L^1(\mathbb{R}^n)$ by assumption, by the dominated convergence theorem we conclude that

$$I_\varepsilon(x) \rightarrow \int_{\mathbb{R}^n} \widehat{f}(y) e^{i\langle x, y \rangle} dy \quad \forall x \in \mathbb{R}^n.$$

On the other hand, applying the previous lemma to $g(y) = \widehat{\varphi^\varepsilon}(y) e^{i\langle x, y \rangle}$

(with x fixed) we have

$$I_\varepsilon(x) = (2\pi)^n \int_{\mathbb{R}^n} g(y) \widehat{f}(y) dy \\ = (2\pi)^n \int_{\mathbb{R}^n} \widehat{g}(y) f(y) dy \\ = (2\pi)^n \int_{\mathbb{R}^n} \widehat{\varphi^\varepsilon}(y-x) f(y) dy \\ = \int_{\mathbb{R}^n} \varphi^\varepsilon(y-x) f(y) dy \\ = (\varphi^\varepsilon * f)(x).$$

Therefore $I_\varepsilon = \varphi^\varepsilon * f \xrightarrow{\varepsilon \rightarrow 0} f$ in $L^1(\mathbb{R}^n)$.

In particular $I_{\nu_n} \xrightarrow{n \rightarrow \infty} f$ in $L^1(\mathbb{R}^n)$,

hence $I_{\nu_{n_k}} \xrightarrow{n \rightarrow \infty} f$ ~~for some subsequence~~ a.e. for some subsequence.

It follows that for a.e. x we have

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(y) e^{i\langle x, y \rangle} dy.$$

□

Corollary

If $f \in L^1(\mathbb{R}^n)$ and $\hat{f} = 0$, then $f = 0$ (a.e.)