

Recall that a linear operator $U: H \rightarrow K$ between Hilbert spaces is called an isometry if

$$\|Ux\| = \|x\| \quad \forall x \in H.$$

Equivalently, by the polarization identity,
 $(Ux, Uy) = (x, y) \quad \forall x, y \in H.$

If U is in addition surjective, then it is called a unitary.

Theorem (Plancherel)

There is a unique unitary $U: L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$
s.t. $Uf = (2\pi)^{\frac{n}{2}} \hat{f}$ for all $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$

Proof

Consider the subspace

$V = \{f \in L^2(\mathbb{R}^n) : f \in L^1(\mathbb{R}^n), \hat{f} \in L^1(\mathbb{R}^n)\}$
of $L^2(\mathbb{R}^n)$.

It $f \in V$, then $\hat{f} \in L^2(\mathbb{R}^n)$, since $\|\hat{f}\|^2 \leq \|f\|_\infty \|f\|$. ③

Moreover, by the lemma used in the proof of Fourier inversion, we have

$$\begin{aligned} \|\hat{f}\|_2^2 &= \int_{\mathbb{R}^n} \hat{f}(x) \overline{\hat{f}}(x) dx = \int_{\mathbb{R}^n} f(x) \overline{\hat{\hat{f}}}(x) dx \\ &= \int_{\mathbb{R}^n} f(x) \overline{\hat{f}}(-x) dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(x) \overline{f}(x) dx \\ &= \frac{1}{(2\pi)^n} \|f\|_2^2. \end{aligned}$$

This shows that we have an isometry

$$U: V \rightarrow L^2(\mathbb{R}^n), \quad Uf = (2\pi)^{n/2} \hat{f}.$$

The space V is dense in $L^2(\mathbb{R}^n)$, since it contains functions $\varphi^\varepsilon * f$ for $f \in C_c(\mathbb{R}^n)$ and $\varphi(x) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{|x|^2}{2}}$, and these functions are dense in $L^2(\mathbb{R}^n)$, because $\varphi^\varepsilon * f \rightarrow f$ and $C_c(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$.

It follows that U extends to an isometric operator ~~$U: L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$~~ $U: L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$. see p. 4

Next, we claim that $Uf = (2\pi)^{n/2} \hat{f}$ for all $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Indeed, for such f , we have $\varphi^\varepsilon * f \in V$ and $\varphi^\varepsilon * f \rightarrow f$ in $L^2(\mathbb{R}^n)$. Hence, on the one hand,

$$U(\varphi^\varepsilon * f) \xrightarrow{\varepsilon \downarrow 0} Uf \text{ in } L^2(\mathbb{R}^n).$$

On the other hand, for all $x \in \mathbb{R}^n$,

$$\begin{aligned} (U(\varphi^\varepsilon * f))(x) &= (2\pi)^{\frac{3n}{2}} \widehat{\varphi^\varepsilon}(x) \widehat{f}(x) \\ &= (2\pi)^{\frac{n}{2}} e^{-\frac{\varepsilon^2 |x|^2}{2}} \widehat{f}(x). \end{aligned}$$

We see that $U(\varphi^\varepsilon * f) \xrightarrow{\varepsilon \downarrow 0} (2\pi)^{\frac{n}{2}} \widehat{f}$ pointwise and in $L^2(\mathbb{R}^n)$. Hence $Uf = (2\pi)^{\frac{n}{2}} \widehat{f}$.

It remains to check that U is surjective.

If $f \in V$, then $Uf \in V \subset L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, so

$$U^2 f = (2\pi)^{\frac{n}{2}} U \widehat{f} = (2\pi)^n \widehat{\widehat{f}},$$

hence $(U^2 f)(x) = f(-x)$ for a.e. x . Therefore

$$\text{Im } U \supset \text{Im } U^2 \supset V.$$

As U is isometric and $L^2(\mathbb{R}^n)$ is see p. 4 complete, $\text{Im } U$ is closed.

As V is dense in $L^2(\mathbb{R}^n)$, we conclude that $\text{Im } U = L^2(\mathbb{R}^n)$. □

For $f \in L^2(\mathbb{R}^n)$ we now define its Fourier transform as the function $\widehat{f} \in L^2(\mathbb{R}^n)$ given by

$$\widehat{f} = (2\pi)^{-\frac{n}{2}} Uf.$$

If $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, we therefore have

$$\widehat{f}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(y) e^{-i\langle x, y \rangle} dy,$$

but if $f \in L^2(\mathbb{R}^n) \setminus L^1(\mathbb{R}^n)$, this integral does not make any sense.

In the proof we used the following two properties:

④

1) If H, K are Hilbert spaces, $V \subset H$ is a dense subspace and $U: V \rightarrow K$ is an isometry, then U extends uniquely to an isometry $U: H \rightarrow K$.

Namely, if $x \in H$, find a sequence $(x_n)_n$ in V converging to x . Then $(x_n)_n$ is Cauchy. Hence $(Ux_n)_n$ is also Cauchy, as

$$\|Ux_n - Ux_m\| = \|x_n - x_m\| \xrightarrow{n, m \rightarrow \infty} 0.$$

As K is complete, $(Ux_n)_n$ is convergent. We then define

$$Ux = \lim_{n \rightarrow \infty} Ux_n.$$

It is not difficult to check that U is well-defined, linear and isometric.

2) If H, K are Hilbert spaces and $U: H \rightarrow K$ is an isometry, then $\text{Im } U$ is closed in K .

Indeed, assume $y \in \overline{\text{Im } U}$, so there exist $x_n \in H$ s.t. $y = \lim_{n \rightarrow \infty} Ux_n$. Then $(Ux_n)_n$ is a Cauchy sequence, hence $(x_n)_n$ is a Cauchy sequence as well, as $\|x_n - x_m\| = \|Ux_n - Ux_m\|$.

Hence $x_n \rightarrow x \in H$ for some x .

Then $Ux = \lim_{n \rightarrow \infty} Ux_n = y$, so $y \in \text{Im } U$.

Schwartz space

①

Proposition

Assume $f \in L^1(\mathbb{R}^n)$ and $x_j f \in L^1(\mathbb{R}^n)$ for some $1 \leq j \leq n$. Then

$$\widehat{\partial_j f} = -i \widehat{x_j f} \quad (\partial_j = \frac{\partial}{\partial x_j})$$

(here by $x_j f$ we mean the function $x \mapsto x_j f(x)$.)

Proof

Follows from the theorem on differentiation of parameter-dependent integrals. $\square$
we have a "dual" result:

Proposition

Assume $f \in C(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ is such that $\partial_j f \in L^1(\mathbb{R}^n)$. Then

$$\widehat{\partial_j f} = i x_j \widehat{f}$$

Proof

Assume first that f is in addition compactly supported. Then by integration by parts we have

$$\int_{\mathbb{R}} \frac{\partial f}{\partial x_j}(y) e^{-i\langle x, y \rangle} dy_j = i x_j \int_{\mathbb{R}} f(y) e^{-i\langle x, y \rangle} dy_j$$

Hence

$$\int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(y) e^{-i\langle x, y \rangle} dy = i x_j \int_{\mathbb{R}^n} f(y) e^{-i\langle x, y \rangle} dy,$$

$$\text{so } \widehat{\partial_j f} = i x_j \widehat{f}.$$

For general f , take any function $\varphi \in C_c^\infty(\mathbb{R}^n)$ s.t. $\varphi(0) = 1$. Define

$$\varphi_k(x) = \varphi\left(\frac{x}{k}\right).$$

Then $\varphi_k f \in C_c(\mathbb{R}^n)$ and $\varphi_k f \xrightarrow[k \rightarrow \infty]{} f$ pointwise and in $L^1(\mathbb{R}^n)$.

We also have

$$\partial_j(\varphi_k f)(x) = \frac{1}{k} (\partial_j \varphi)\left(\frac{x}{k}\right) f(x) + \varphi_k(x) (\partial_j f)(x)$$

Since $|(\partial_j \varphi)(\frac{x}{k})| \leq \|\partial_j \varphi\|_\infty$, we conclude that $\partial_j(\varphi_k f) \rightarrow \partial_j f$ pointwise and in $L^1(\mathbb{R}^n)$. Hence

$$\widehat{\partial_j(\varphi_k f)} \rightarrow \widehat{\partial_j f}$$

uniformly on $\mathbb{R}^n$. On the other hand, by the case we started with, we have

$$\widehat{\partial_j(\varphi_k f)}(x) = i x_j \widehat{\varphi_k f}(x)$$

and $\widehat{\varphi_k f} \rightarrow \widehat{f}$ uniformly on $\mathbb{R}^n$. Hence

$$\widehat{\partial_j f}(x) = i x_j \widehat{f}(x).$$

□