

Corollary

If $f, \partial_j f \in L^1(\mathbb{R})$, then $x_j \hat{f}(x) \xrightarrow{x \rightarrow \infty} 0$.

Proof

Indeed, $x_j \hat{f} = -i \widehat{\partial_j f} \in C_0(\mathbb{R}^n)$ by the Riemann-Lebesgue lemma. (28)

By applying the previous two propositions repeatedly we get the following:

Corollary

1) If $x^\alpha f \in L^1(\mathbb{R}^n)$ for all $|\alpha| \leq N$ then $\hat{f} \in C^N(\mathbb{R}^n)$ and $\partial^\alpha \hat{f} = (-i)^{|\alpha|} \widehat{x^\alpha f}$.

2) If $f \in C^N(\mathbb{R}^n)$ and $\partial^\alpha f \in L^1(\mathbb{R}^n)$ for all $|\alpha| \leq N$, then $\widehat{\partial^\alpha f} = i^{|\alpha|} x^\alpha \hat{f}$ and hence $(1+|x|)^N \hat{f}(x) \xrightarrow{x \rightarrow \infty} 0$.

here $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, $|\alpha| = \alpha_1 + \dots + \alpha_n$,
 $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$.

Def

A function f on $\mathbb{R}^n$ is called a Schwartz function if $f \in C^\infty(\mathbb{R}^n)$ and $x^\alpha \partial^\beta f$ is bounded for all $\alpha, \beta \in \mathbb{Z}_+^n$. The space $S(\mathbb{R}^n)$ of Schwartz functions is called the Schwartz space.

Note that for every $f \in C^\infty(\mathbb{R}^n)$ the following conditions are equivalent:

- 1) $x^\alpha \partial^\beta f$ is bounded $\forall \alpha, \beta$;
- 2) $x^\alpha (\partial^\beta f)(x) \xrightarrow{x \rightarrow \infty} 0 \quad \forall \alpha, \beta$;
- 3) $(1+|x|)^N \partial^\beta f \xrightarrow{x \rightarrow \infty} 0 \quad \forall N \geq 1 \quad \forall \beta \in \mathbb{Z}_+^n$.

Example

$C_c^\infty(\mathbb{R}^n) \subset S(\mathbb{R}^n)$, $e^{-a|x|^2} \in S(\mathbb{R}^n)$ ($a > 0$).

If $f \in S(\mathbb{R}^n)$, then $x^\alpha \partial^\beta f \in S(\mathbb{R}^n)$.

If $f, g \in S(\mathbb{R}^n)$, then $fg \in S(\mathbb{R}^n)$.

Clearly, $S(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) \quad \forall 1 \leq p \leq \infty$
 As $C_c^\infty(\mathbb{R}^n) \subset S(\mathbb{R}^n)$, $S(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$.

Theorem
 The Fourier transform maps $S(\mathbb{R}^n)$ onto $S(\mathbb{R}^n)$.

Proof
 If $f \in S(\mathbb{R}^n)$, then $\hat{f} \in S(\mathbb{R}^n)$ by the previous corollary.

If $g \in S(\mathbb{R}^n)$, then $\check{g} \in S(\mathbb{R}^n)$, where $\check{g}(x) = g(-x)$. By the Fourier inversion formula we have $\hat{\check{g}}(x) = \check{g}(-x) = g(x)$, so $f = \hat{g}$ for $g = \check{\check{f}} \in S(\mathbb{R}^n)$. $\square$

Conditional expectations

Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space.
(In probability theory the measure is usually denoted by $\mathbb{P}$.)

A measurable function $X: \Omega \rightarrow \mathbb{R}$ is called a random variable

If $\int_{\Omega} |X| d\mathbb{P} < \infty$, then $\mathbb{E}(X) \stackrel{\text{def}}{=} \int_{\Omega} X d\mathbb{P}$

is called the (mathematical) expectation, or the expected value, of X .

Fact

Assume $((\Omega_i, \mathcal{B}_i, \mathbb{P}_i))_{i \in I}$ is a collection of probability spaces. Consider $\Omega = \prod_i \Omega_i$ and $\mathcal{B} = \prod_i \mathcal{B}_i$, that is, $\mathcal{B}$ is the σ -algebra generated by sets of the form $\prod_i A_i$, where $A_i \in \mathcal{B}_i$ and $A_i = \Omega_i$ for all but finitely many i 's. Then there exists a unique probability measure $\mathbb{P}$ on $(\Omega, \mathcal{B})$ s.t.

$$\mathbb{P}\left(\prod_i A_i\right) = \prod_i \mathbb{P}_i(A_i)$$

for all $A_i \in \mathcal{B}_i$ s.t. $A_i = \Omega_i$ for all but finitely many i 's.

Remark

Assume $(\Omega_n)_{n=1}^\infty$ is a sequence of separable metric spaces of diameter ≤ 1 .

Then $\Omega = \prod_{n=1}^\infty \Omega_n$ is a metric space with metric $d(x, y) = \sum_{n=1}^\infty \frac{1}{2^n} d_n(x_n, y_n)$.

If $x = (x_n)_n \in \Omega$, then $B_{r_1}(x_1) \times \dots \times B_{r_n}(x_n) \times \Omega_{n+1} \times \dots$ is open in $\Omega \forall r_1, \dots, r_n > 0$, and any neighbourhood of x contains such an open set.

It follows that $\prod_{n=1}^\infty \mathcal{B}(\Omega_n)$ coincides with the Borel σ -algebra $\mathcal{B}(\Omega)$.

Assume μ_n is a Borel probability measure on Ω_n . Then the existence of $\mu = \prod_{n=1}^\infty \mu_n$ can be deduced from the Riesz-Markov theorem, if we assume that Ω_n are also compact. Indeed, then Ω is compact as well. By the Stone-Weierstrass theorem the functions of the form $f(x) = f_1(x_1) \dots f_n(x_n)$ ($f_i \in C(\Omega_i)$) span a dense subspace of $C(\Omega)$. Since the measures $\mu_1 \times \dots \times \mu_n$ exist, we get a well-defined positive linear functional on such a set. For $f(x) = f_1(x_1) \dots f_n(x_n)$ we have

$$\mu(f) = \prod_{k=1}^\infty \int_{\Omega_k} f_k d\mu_k.$$

By continuity it extends to $C(\Omega)$, hence defines a Borel measure $\mathbb{P}$ by the Riesz-Markov theorem.

Example (coin-flipping)

$$\Omega = \prod_{n=1}^{\infty} \{0,1\}, \mathcal{B} = \prod_{n=1}^{\infty} \mathcal{P}(\{0,1\}), \mathbb{P} = \prod_{n=1}^{\infty} \left(\frac{1}{2} \delta_0 + \frac{1}{2} \delta_1 \right)$$

In other words,

$$\mathbb{P}(\{x : x_1 = a_1, \dots, x_n = a_n\}) = \frac{1}{2^n}$$

for all $n \geq 1$ and $a_1, \dots, a_n \in \{0,1\}$.

$$X_n: \Omega \rightarrow \{0,1\}, X_n(x) \stackrel{\text{def}}{=} x_n.$$

Assume X, Y are random variables on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$. One is often interested in expectations of X under conditions on Y . Formally, a conditional expectation is defined by

$$\mathbb{E}(X | Y \in A) \stackrel{\text{def}}{=} \frac{\int_{Y^{-1}(A)} X d\mathbb{P}}{\mathbb{P}(Y^{-1}(A))}$$

for Borel sets $A \subset \mathbb{R}$.

More generally (and formally), assume $\mathcal{F} \subset \mathcal{B}$ is a σ -algebra.

If $X \in L^1(\Omega, \mathcal{B}, d\mathbb{P})$, we can consider a complex measure $\mathbb{P}_X$ on $(\Omega, \mathcal{B})$:

$$\mathbb{P}_X(A) = \int_A X d\mathbb{P}.$$

Then $\mathbb{P}_X \ll \mathbb{P}$, hence also $\mathbb{P}_X|_{\mathcal{F}} \ll \mathbb{P}|_{\mathcal{F}}$.

Def

The conditional expectation of X given $\mathcal{F}$

is defined by

$$\mathbb{E}(X|\mathcal{F}) \stackrel{\text{def}}{=} \frac{d(\mathbb{P}_X|_{\mathcal{F}})}{d(\mathbb{P}|_{\mathcal{F}})} \in L^1(\Omega, \mathcal{F}, d\mathbb{P}|_{\mathcal{F}}).$$

In other words, $\mathbb{E}(X|\mathcal{F})$ is the unique element of $L^1(\Omega, \mathcal{F}, d\mathbb{P}|_{\mathcal{F}})$ s.t.

$$\int_A X d\mathbb{P} = \int_A \mathbb{E}(X|\mathcal{F}) d\mathbb{P} \quad \forall A \in \mathcal{F}.$$

Example

Assume we are given a finite or countable partition of Ω into measurable sets A_n , $\mathcal{F} = \sigma((A_n)_n)$ (so the elements of $\mathcal{F}$ are unions of some of the sets A_n). Then

$$\mathbb{E}(X|\mathcal{F}) = \sum_n \frac{\int_{A_n} X d\mathbb{P}}{\mathbb{P}(A_n)} \mathbb{1}_{A_n}.$$

The following properties are immediate:

(1) $\mathbb{E}(\cdot|\mathcal{F}) : L^1(\Omega, \mathcal{B}, d\mathbb{P}) \rightarrow L^1(\Omega, \mathcal{F}, d\mathbb{P}|_{\mathcal{F}})$ is a linear operator;

(2) if $X \geq 0$, then $\mathbb{E}(X|\mathcal{F}) \geq 0$;

(3) if X is $\mathcal{F}$ -measurable, then $\mathbb{E}(X|\mathcal{F}) = X$.