

Example (Bernoulli shifts)

Assume $(X, \mathcal{A}, \nu)$ is a probability space.

Define $\Omega = \prod_{n=1}^{\infty} X$, $\mathcal{B} = \bigotimes_{n=1}^{\infty} \mathcal{A}$, $P = \prod_{n=1}^{\infty} \nu$.

Define $T: \Omega \rightarrow \Omega$ by $T(x_1, x_2, \dots) = (x_2, x_3, \dots)$.

Then T is measurable, P -preserving.

We claim that T is mixing, hence ergodic.

Let $\mathcal{F}_n$ be the σ -algebra generated by the sets $A_1 \times \dots \times A_n \times X \times X \times \dots$. Then, by definition, $\mathcal{B} = \sigma(\bigcup_{n=1}^{\infty} \mathcal{F}_n)$.

If $A, B \in \mathcal{F}_{n_0}$ for some n_0 , then

$$P(T^{-n}(A) \cap B) = P(A)P(B) \quad \forall n \geq n_0.$$

For general $A, B \in \mathcal{B}$ we can take $\epsilon > 0$ and

find $\tilde{A}, \tilde{B} \in \mathcal{F}_{n_0}$ for n_0 large enough s.t.

$P(A \Delta \tilde{A}) < \epsilon$, $P(B \Delta \tilde{B}) < \epsilon$. Then

$$\begin{aligned} |P(T^{-n}(A) \cap B) - P(A)P(B)| \\ \leq 4\epsilon + |P(T^{-n}(\tilde{A}) \cap \tilde{B}) - P(\tilde{A})P(\tilde{B})| \\ = 4\epsilon \text{ for } n \text{ large enough.} \end{aligned}$$

Hence $P(T^{-n}(A) \cap B) \rightarrow P(A)P(B)$.

By Birkhoff's theorem we conclude now that $\forall f \in L^1(\Omega, \mathcal{B}, dP)$ we have

$$\frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k \rightarrow \int_{\Omega} f dP \quad \text{a.e.}$$

In particular, if X is finite and/or countable and $\mathcal{A} = \mathcal{P}(X)$, fix

$a_1, \dots, a_m \in X$ and consider

$\mathcal{I} = \prod_{\mathcal{R}(a_1, \dots, a_m)} \mathcal{I}$, where $\mathcal{R}(a_1, \dots, a_m) = \{x \in \mathcal{R} :$

$x_1 = a_1, \dots, x_m = a_m\}$. Then

$$(\mathcal{I} \circ T^k)(x) = \begin{cases} 1, & \text{if } x_{k+1} = a_1, \dots, x_{k+m} = a_m, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore we get that for $\mathbb{P}$ -a.e. x we have

$$\frac{|\{0 \leq k \leq n-1 : x_{k+1} = a_1, \dots, x_{k+m} = a_m\}|}{n} \xrightarrow{n \rightarrow \infty} \nu(a_1) \dots \nu(a_m)$$

This is called the infinite monkey theorem.

Def

Assume $\mathcal{I}$ is a compact metric space, $T: \mathcal{I} \rightarrow \mathcal{I}$ is a continuous map. We say that T is uniquely ergodic if there exists a unique T -invariant Borel probability measure on $\mathcal{I}$.

Lemma

If T is uniquely ergodic and $\mathbb{P}$ is the unique T -invariant Borel probability measure, then T is an ergodic transformation of $(\mathcal{I}, \mathcal{B}(\mathcal{I}), \mathbb{P})$.

Proof

Assume there is $A \in \mathcal{B}(\mathcal{I})^T$ s.t.

$P(A) \neq 0, 1$. Consider the Borel probability measure $\mu(B) = \frac{P(A \cap B)}{P(A)}$.

Then μ is T -invariant and $\mu \neq P$, contradicting unique ergodicity. $\square$

Fact

Assume Ω is a compact metric space,

$T: \Omega \rightarrow \Omega$ is a continuous map.

Then the following conditions are equivalent:

(1) T is uniquely ergodic;

(2) for every $f \in C(\Omega)$, the sequence

$\left(\frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) \right)_{n=1}^{\infty}$ is Cauchy $\forall x \in \Omega$.

If these conditions are satisfied, then for the unique T -invariant probability measure P on $(\Omega, \mathcal{B}(\Omega))$ we have

$$\frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) \rightarrow \int_{\Omega} f dP \quad \forall f \in C(\Omega), \forall x \in \Omega.$$

(The proof is independent of Birkhoff's theorem and is, in fact, easier.)

Remark/warning

Unique ergodicity does not imply that $\frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) \rightarrow \int_{\Omega} f dP$ for all $f \in L^1(\Omega, dP)$ and all $x \in \Omega$.

Def

Assume Ω is a compact metric space,
 P is a Borel probability measure on Ω .

A sequence $(x_n)_{n=1}^{\infty}$ is said to be
equidistributed in Ω (with respect to P) if

$$\frac{1}{n} \sum_{k=0}^{n-1} f(x_k) \xrightarrow{n \rightarrow \infty} \int_{\Omega} f dP \quad \forall f \in C(\Omega).$$

Therefore if $T: \Omega \rightarrow \Omega$ is a Borel map
leaving P invariant, then

- if T is ergodic, then $(T^n x)_{n=1}^{\infty}$ is
equidistributed in Ω for P -a.e. $x \in \Omega$;
- if T is uniquely ergodic, then $(T^n x)_{n=1}^{\infty}$
is equidistributed in $\Omega \quad \forall x \in \Omega$.

Remark

If $(x_n)_{n=1}^{\infty}$ is equidistributed in Ω w.r.t. P ,
then for any open set $U \subset \Omega$ s.t. $P(\partial U) = 0$
we have
$$\frac{|\{0 \leq k \leq n-1 : x_k \in U\}|}{n} \xrightarrow{n \rightarrow \infty} P(U).$$

This can be checked by applying the

definition of equidistribution to

$f \in C(\Omega)$ s.t. $0 \leq f \leq 1$ and either

$\text{supp } f \subset U$, or $f|_U = 1$ and

$\text{supp } f \subset \{x : d(x, U) < \varepsilon\}$ for some $\varepsilon > 0$.

(The same is true for Borel U s.t. $P(\partial U) = 0$.)

Example (irrational rotations)

We can identify $\{0,1\}$ with $\mathbb{T}$ using the map $t \mapsto e^{2\pi i t}$. This gives a measure $\lambda_{\mathbb{T}}$ on $\mathbb{T}$.

Fix $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and consider $T_{\theta}: \mathbb{T} \rightarrow \mathbb{T}$, $T_{\theta} z = e^{2\pi i \theta} z$. Then $\lambda_{\mathbb{T}}$ is T_{θ} -invariant. We claim that T_{θ} is uniquely ergodic.

Indeed, assume μ is a T_{θ} -invariant Borel probability measure on $\mathbb{T}$.

Consider the functions $e_n \in C(\mathbb{T})$,

$e_n(z) = z^n$ ($n \in \mathbb{Z}$). Then

$e_n \circ T_{\theta} = e^{2\pi i n \theta} e_n$, so

$$\int_{\mathbb{T}} e_n d\mu = \int_{\mathbb{T}} e_n \circ T_{\theta} d\mu = e^{2\pi i n \theta} \int_{\mathbb{T}} e_n d\mu.$$

Hence $\int_{\mathbb{T}} e_n d\mu = 0 = \int_{\mathbb{T}} e_n d\lambda_{\mathbb{T}}$ for $n \neq 0$.

As $\text{span}\{e_n : n \in \mathbb{Z}\}$ is dense in $C(\mathbb{T})$ by Weierstrass' theorem, we conclude that

$$\mu = \lambda_{\mathbb{T}}.$$

Conclusion: for every $z \in \mathbb{T}$, the sequence $(e^{2\pi i n \theta} z)_{n=1}^{\infty}$ is equidistributed in $\mathbb{T}$.

Equivalently, for every $t \in [0,1]$, the sequence $(\{t + n\theta\})_{n=1}^{\infty}$ is equidistributed.

Fractional part function

on $[0,1]$ with respect to the Lebesgue measure, so

$$\frac{|\{1 \leq k \leq n; \{t + k\theta\} \in (a,b)\}|}{n} \xrightarrow{n \rightarrow \infty} b-a$$

for all $0 \leq a < b \leq 1$.

With more work one can prove the following.

Fact (Weyl)

Assume $p(x) = a_k x^k + \dots + a_1 x + a_0$ is a real polynomial of degree ≥ 1 s.t. at least one coefficient among $a_1, \dots, a_k$ is irrational. Then the sequence $(\{p(n)\})_{n=1}^{\infty}$ is equidistributed on $[0,1]$.