

Compulsory assignment in MAT3400/4400, Spring 2026

Problem 1. Let $(X, \mathcal{B}, \mu)$ be a measure space. A measurable subset $A \subset X$ with $\mu(A) > 0$ is called an *atom* of the measure μ if whenever $B \subset A$ is measurable, we must have $\mu(B) = 0$ or $\mu(A \setminus B) = 0$. We say that μ is *atomless*, or *diffuse*, if it has no atoms.

(a) Assume μ is σ -finite. Show that X decomposes into a disjoint union of at most countably many measurable sets X_n ($n = 0, \dots$) such that the restriction of μ to X_0 is atomless, while every set X_n with $n \geq 1$ is an atom of μ of finite measure.

(b) Assume μ is σ -finite and the σ -algebra $\mathcal{B}$ is generated by countably many sets A_n that separate points of X , meaning that for any points $x \neq y$ there is A_n containing one of them and not containing the other (for example, the σ -algebra of Borel subsets of a separable metric space has this property). Show that $\{y\} \in \mathcal{B}$ for all $y \in X$ and, for any atom A of μ , there is a unique point $x \in A$ such that $\mu(\{x\}) > 0$.

In this case it is then common to call the points x with $\mu(\{x\}) > 0$ the atoms of μ rather than more general sets as defined above.

(c) Assume μ is a finite atomless measure. Show that for every $A \in \mathcal{B}$ with $\mu(A) > 0$ and every $\varepsilon > 0$ there exists a measurable subset $B \subset A$ such that $0 < \mu(B) < \varepsilon$.

(d) Assume that μ is an atomless probability measure. Show that there exist measurable sets A_t for $t \in [0, 1]$ such that $\mu(A_t) = t$ and $A_s \subset A_t$ for $s \leq t$.

Hint: use (c) to construct first $A_{1/2}$ and then A_t for dyadic rationals $t \in [0, 1]$.

(e) Assume again that μ is an atomless probability measure. Use (d) to define a measurable function $f: X \rightarrow [0, 1]$ such that $\mu(f^{-1}([0, t])) = t$ for all $t \in [0, 1]$. Conclude that for any Borel subset $A \subset [0, 1]$ we have $\mu(f^{-1}(A)) = \lambda(A)$, where λ is the Lebesgue measure.

Problem 2. Assume X is a metric space and μ is a regular Borel measure on X (recall that regularity is automatic if, for example, X is a complete separable metric space and μ is finite on every ball).

(a) Show that there exists a largest open set $U \subset X$ such that $\mu(U) = 0$. The closed set $X \setminus U$ is called the *support* of μ and is denoted by $\text{supp } \mu$.

(b) Recall that the support $\text{supp } f$ of a continuous function f on X is the closure of the set $\{x : f(x) \neq 0\}$. Show that $\text{supp } \mu$ is the smallest closed set with the following property: if $f: X \rightarrow [0, +\infty)$ is continuous and $\text{supp } \mu \cap \text{supp } f = \emptyset$, then $\int_X f d\mu = 0$. Show also that $\text{supp } \mu = X$ if and only if $\int_X f d\mu > 0$ for every nonzero continuous function $f: X \rightarrow [0, +\infty)$.

(c) We say that μ is *concentrated* on a Borel set $A \subset X$ if $\mu(X \setminus A) = 0$. To illustrate the difference between the support and a set of concentration of a measure, give an example of a Borel probability measure on $[0, 1]$ that is concentrated on a countable set, but its support is the entire interval $[0, 1]$.

Problem 3. Assume A is a nonempty bounded subset of a Hilbert space H . Let r be the infimum of the radii of closed balls containing A , that is,

$$r = \inf\{s \geq 0 \mid \text{there exists } x \in H \text{ such that } \|y - x\| \leq s \text{ for all } y \in A\}.$$

(a) Show that there exists a unique closed ball of radius r containing A . The center of this ball is called the *circumcenter* of A .

Hint: recall how the existence of a point in a closed convex set closest to a given point is proved.

(b) Show that if A is convex and closed, then the circumcenter of A lies in A .