

Solution to compulsory assignment in MAT3400/4400, Spring 2026

Problem 1. Let $(X, \mathcal{B}, \mu)$ be a measure space.

(a) Assume μ is σ -finite. Show that X decomposes into a disjoint union of at most countably many measurable sets X_n ($n = 0, \dots$) such that the restriction of μ to X_0 is atomless, while every set X_n with $n \geq 1$ is an atom of μ of finite measure.

Solution. Since X decomposes into a disjoint union of countably many sets of finite measure, it suffices to consider the case of finite measure. Assume that μ has atoms, as otherwise there is nothing to prove. Let us say that two atoms A and B are equivalent if $\mu(A \cap B) > 0$; equivalently, $\mu(A \setminus B) = \mu(B \setminus A) = 0$. Let I be the set of equivalence classes of atoms, and for every $i \in I$ choose a representative A_i .

If two atoms A and B are not equivalent, then $\mu(A \cap B) = 0$. It follows that if $i_1, \dots, i_n$ are different equivalence classes, then

$$\mu(A_{i_1}) + \dots + \mu(A_{i_n}) = \mu(\cup_{k=1}^n A_{i_k}) \leq \mu(X).$$

This means that $\sum_{i \in I} \mu(A_i) \leq \mu(X) < \infty$. As we know, this implies that the set I is countable.

Enumerate the elements of I as $i_1, i_2, \dots$. Then let $X_n = A_{i_n} \setminus (A_{i_1} \cup \dots \cup A_{i_{n-1}})$ for $n \geq 1$, and $X_0 = X \setminus (X_1 \cup X_2 \cup \dots)$. Every set X_n for $n \geq 1$ is an atom equivalent to A_{i_n} . Every atom A is equivalent to one of the atoms X_n ($n \geq 1$), hence $\mu(A \cap X_0) = 0$, so there are no atoms contained in X_0 .

(b) Assume μ is σ -finite and the σ -algebra $\mathcal{B}$ is generated by countably many sets A_n that separate points of X . Show that $\{y\} \in \mathcal{B}$ for all $y \in X$ and, for any atom A of μ , there is a unique point $x \in A$ such that $\mu(\{x\}) > 0$.

Solution. Take $y \in X$. For every n , put $B_n = A_n$ if $y \in A_n$ and $B_n = X \setminus A_n$ if $y \notin A_n$. Then $\cap_{n=1}^{\infty} B_n = \{y\}$, since the sets A_n separate points of X . Thus, $\{y\} \in \mathcal{B}$.

If A is an atom of μ , for every n , put $B_n = A_n \cap A$ if $\mu(A_n \cap A) > 0$ and $B_n = A \setminus A_n$ if $\mu(A_n \cap A) = 0$. Then, on the one hand, $\cap_{n=1}^{\infty} B_n$ consists of at most one point. On the other hand, since $\mu(B_1 \cap \dots \cap B_n) = \mu(A)$ for all n , we have

$$\mu(B) = \lim_{n \rightarrow \infty} \mu(B_1 \cap \dots \cap B_n) = \mu(A) > 0.$$

It follows that $B = \{x\}$ for some $x \in A$, and $\mu(\{x\}) = \mu(A) > 0$. Then $\mu(A \setminus \{x\}) = 0$, in particular, $\mu(\{y\}) = 0$ for all $y \in A \setminus \{x\}$. Hence x is the unique point in A with $\mu(\{x\}) > 0$.

(c) Assume μ is a finite atomless measure. Show that for every $A \in \mathcal{B}$ with $\mu(A) > 0$ and every $\varepsilon > 0$ there exists a measurable subset $B \subset A$ such that $0 < \mu(B) < \varepsilon$.

Solution. Since A is not an atom, there exists a measurable subset $C \subset A$ such that $0 < \mu(C) < \mu(A)$. Then either $\mu(C) \leq \mu(A)/2$ or $\mu(A \setminus C) \leq \mu(A)/2$. In the first case put $A_1 = C$, in the second case put $A_1 = A \setminus C$. Then apply the same argument to A_1 , and so on. This way we inductively construct a decreasing sequence of subsets A_n of A such that $0 < \mu(A_n) \leq \mu(A)/2^n$. Choosing n large enough we get $\mu(A_n) < \varepsilon$ and then can take $B = A_n$.

(d) Assume that μ is an atomless probability measure. Show that there exist measurable sets A_t for $t \in [0, 1]$ such that $\mu(A_t) = t$ and $A_s \subset A_t$ for $s \leq t$.

Solution. Put $A_0 = \emptyset$ and $A_1 = X$. Following the hint, let us prove that there is $A \in \mathcal{B}$ such that $\mu(A) = 1/2$. For this consider the set I of collections $\mathcal{C}$ of measurable subsets

of X satisfying the following properties: the elements $B \in \mathcal{C}$ are mutually disjoint, $\mu(B) > 0$, and $\sum_{B \in \mathcal{C}} \mu(B) \leq 1/2$. Let us partially order the set I by inclusion. Then every chain $(\mathcal{C}_j)_{j \in J}$ in I has an upper bound, namely, $\cup_{j \in J} \mathcal{C}_j$. By Zorn's lemma we conclude that there is a maximal element $\mathcal{C} \in I$. As in (a), the condition $\sum_{B \in \mathcal{C}} \mu(B) \leq 1/2 < \infty$ implies that $\mathcal{C}$ is countable. Hence the set $A = \cup_{B \in \mathcal{C}} B$ is measurable and $\mu(A) \leq 1/2$. If the inequality is strict, we can use (c) to find a measurable subset $B \subset X \setminus A$ such that $0 < \mu(B) < \frac{1}{2} - \mu(A)$. But then $\mathcal{C} \cup \{B\} \in I$ is strictly larger than $\mathcal{C}$, contradicting the maximality of $\mathcal{C}$. Hence $\mu(A) = 1/2$. We let $A_{1/2} = A$.

Once $A_{1/2}$ is constructed, we can apply the same arguments to this set and its complement (and the corresponding restrictions of μ multiplied by 2). We then get a subset $A_{1/4} \subset A_{1/2}$ and a subset $A \subset A_1 \setminus A_{1/2}$ of measure $1/4$ each. We put $A_{3/4} = A_{1/2} \cup A$, so now we have the sets $A_{k/4}$ for all $0 \leq k \leq 4$. Then we divide the sets $A_{k/4} \setminus A_{(k-1)/4}$ into two halves and add one of them to $A_{(k-1)/4}$ to get $A_{(2k-1)/8}$ for $1 \leq k \leq 4$, so now we have the sets $A_{k/8}$ for all $0 \leq k \leq 8$. By continuing this procedure we can construct A_t for all dyadic rationals $t = k/2^n$ ($0 \leq k \leq 2^n$) in $[0, 1]$. Let us denote by $D \subset [0, 1]$ the set of such numbers. For the remaining t we can then put

$$A_t = \bigcap_{\substack{s \in D \\ t \leq s}} A_s,$$

as then $\mu(A_t) = \lim_{D \ni s \downarrow t} \mu(A_s) = t$.

(e) Assume again that μ is an atomless probability measure. Use (d) to define a measurable function $f: X \rightarrow [0, 1]$ such that $\mu(f^{-1}([0, t])) = t$ for all $t \in [0, 1]$. Conclude that for any Borel subset $A \subset [0, 1]$ we have $\mu(f^{-1}(A)) = \lambda(A)$, where λ is the Lebesgue measure.

Solution. Let A_t and D be as in (d). Define $f: X \rightarrow [0, 1]$ by

$$f(x) = \inf\{t \in [0, 1] : x \in A_t\}.$$

Then $f(x) < t$ if and only if $x \in A_s$ for some $s < t$, that is,

$$f^{-1}([0, t)) = \bigcup_{s < t} A_s = \bigcup_{\substack{s \in D \\ s < t}} A_s.$$

The second equality shows that $f^{-1}([0, t))$ is measurable. Since the sets $[0, t)$ generate the Borel σ -algebra on $[0, 1]$, it follows that f is measurable.

Consider the Borel measure $\nu = f_*\mu$ on $[0, 1]$, that is, $\nu(A) = \mu(f^{-1}(A))$. By construction we have $\nu([0, t)) = \lim_{D \ni s \uparrow t} \mu(A_s) = t$ and $\nu([0, 1]) = 1$. Since $[s, t) = [0, t) \setminus [0, s)$, we then also get $\nu([s, t)) = t - s = \lambda([s, t))$ for all $0 \leq s < t \leq 1$. As we know, the Lebesgue measure is completely determined by its values on the intervals $[s, t)$. It follows that $\nu(A) = \lambda(A)$ for all Borel subsets $A \subset [0, 1]$.

Problem 2. Assume X is a metric space and μ is a regular Borel measure on X .

(a) Show that there exists a largest open set $U \subset X$ such that $\mu(U) = 0$. The closed set $X \setminus U$ is called the *support* of μ and is denoted by $\text{supp } \mu$.

Solution. Consider the union U of all open subsets $V \subset X$ such that $\mu(V) = 0$. We claim that $\mu(U) = 0$. By regularity it suffices to show that if $K \subset U$ is compact, then $\mu(K) = 0$. By compactness, K can be covered by finitely many open sets $V_1, \dots, V_n$ such that $\mu(V_i) = 0$ for all i . Then $\mu(K) \leq \sum_{i=1}^n \mu(V_i) = 0$. This proves our claim. Clearly, U is the largest open set of measure zero.

(b) Show that $\text{supp } \mu$ is the smallest closed set with the following property: if $f: X \rightarrow [0, +\infty)$ is continuous and $\text{supp } \mu \cap \text{supp } f = \emptyset$, then $\int_X f d\mu = 0$. Show also that $\text{supp } \mu = X$ if and only if $\int_X f d\mu > 0$ for every nonzero continuous function $f: X \rightarrow [0, +\infty)$.

Solution. If $\text{supp } \mu \cap \text{supp } f = \emptyset$, then $\int_X f d\mu = \int_{\text{supp } \mu} f d\mu = 0$. On the other hand, if $F \subset \text{supp } \mu$ is a proper closed subset, take any point $x_0 \in \text{supp } \mu \setminus F$ and let

$$r = d(x_0, F) > 0.$$

(If $F = \emptyset$, take any $r > 0$.) Take a positive continuous function f such that $f(x_0) > 0$ and $\text{supp } f \subset B_{r/2}(x_0)$, for example, $f(x) = d(x, X \setminus B_{r/4}(x_0))$. Then $F \cap \text{supp } f = \emptyset$. Consider the open set $V = \{x : f(x) > 0\}$. As $V \not\subset F$, we have $\mu(V) > 0$ and hence $\int_X f d\mu = \int_V f d\mu > 0$. This solves the first part of the problem.

If $\text{supp } \mu = X$ and $f: X \rightarrow [0, +\infty)$ is a nonzero continuous function, then letting $V = \{x : f(x) > 0\} \neq \emptyset$ we get $\mu(V) > 0$ and hence $\int_X f d\mu = \int_V f d\mu > 0$. Conversely, assume $\int_X f d\mu > 0$ for every nonzero continuous function $f: X \rightarrow [0, +\infty)$. Take any nonempty open set V . Then, as in the previous paragraph, we can find a nonzero positive continuous function f with support inside a ball contained by V . Then $\int_V f d\mu = \int_X f d\mu > 0$, hence $\mu(V) > 0$. Therefore the only open set of measure zero is the empty set, that is, $\text{supp } \mu = X$.

(c) We say that μ is *concentrated on* a Borel set $A \subset X$ if $\mu(X \setminus A) = 0$. To illustrate the difference between the support and a set of concentration of a measure, give an example of a Borel probability measure on $[0, 1]$ that is concentrated on a countable set, but its support is the entire interval $[0, 1]$.

Solution. Enumerate the rational numbers in $[0, 1]$ as $x_1, x_2, \dots$. Then let

$$\mu = \sum_{n=1}^{\infty} \frac{1}{2^n} \delta_{x_n}, \quad \text{that is,} \quad \mu(A) = \sum_{n=1}^{\infty} \frac{1}{2^n} \mathbb{1}_A(x_n).$$

Then μ is concentrated on the rational numbers, but $\text{supp } \mu = [0, 1]$, since every nonempty open set $U \subset [0, 1]$ contains some x_n and hence $\mu(U) > 0$.

Problem 3. Assume A is a nonempty bounded subset of a Hilbert space H . Let r be the infimum of the radii of closed balls containing A .

(a) Show that there exists a unique closed ball of radius r containing A . The center of this ball is called the *circumcenter* of A .

Solution. Choose sequences of elements $x_n \in H$ and numbers s_n such that for every n the ball of radius s_n with center x_n contains A and $s_n \rightarrow r$ as $n \rightarrow \infty$. Then $(x_n)_n$ is a Cauchy sequence. Indeed, for any $y \in A$, following the hint and using the parallelogram law we get

$$\begin{aligned} \left\| y - \frac{x_n + x_m}{2} \right\|^2 &= \frac{1}{4} \|(y - x_n) + (y - x_m)\|^2 \\ &= -\frac{1}{4} \|(y - x_n) - (y - x_m)\|^2 + \frac{1}{2} \|y - x_n\|^2 + \frac{1}{2} \|y - x_m\|^2 \\ &\leq -\frac{1}{4} \|x_n - x_m\|^2 + \frac{1}{2} (s_n^2 + s_m^2). \end{aligned}$$

By definition of r the supremum of the left hand side over all $y \in A$ must be at least r^2 . Therefore

$$r^2 \leq -\frac{1}{4} \|x_n - x_m\|^2 + \frac{1}{2} (s_n^2 + s_m^2),$$

or in other words,

$$\|x_n - x_m\|^2 \leq 2(s_n^2 - r^2) + 2(s_m^2 - r^2).$$

Hence $(x_n)_n$ is a Cauchy sequence. Let x be its limit. Passing to the limit in the inequality $\|y - x_n\| \leq s_n$, we get that $\|y - x\| \leq r$ for all $y \in A$.

If x' is another point such that $\|y - x'\| \leq r$ for all $y \in A$, then we could take $x_n = x$ and $x_m = x'$ in the above computation and conclude that $x = x'$.

(b) Show that if A is convex and closed, then the circumcenter of A lies in A .

Solution. Since A is convex and closed, there exists a point $y_0 \in A$ closest to x . Assume $y_0 \neq x$. Then there exists a point $y \in A$ such that $\|y - y_0\| > r$. Then the triangle with vertices x , y_0 and y has the longest edge $[y_0, y]$. But then the closest point on $[y_0, y]$ to x cannot be y_0 , which is a contradiction, as y_0 was assumed to be the closest point in A to x .