

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3400/4400 — Linear Analysis with Applications

Day of examination: Wednesday 11 June 2025

Examination hours: 15:00–19:00

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Please make sure to justify all of your answers.

Problem 1

Let X be a non-empty set and let $(A_n)_{n \in \mathbb{N}} \subset X$ be a sequence of pairwise disjoint sets that satisfies

$$\bigcup_{n \in \mathbb{N}} A_n = X.$$

1a (weight 10%)

Let $\mathcal{P}(\mathbb{N})$ denote the power set of the natural numbers $\mathbb{N}$. Show that $\mathcal{A} = \{\cup_{n \in I} A_n \mid I \in \mathcal{P}(\mathbb{N})\}$ is a σ -algebra on X .

1b (weight 10%)

Let $\mathcal{B}$ be a σ -algebra on X satisfying for every set $B \in \mathcal{B}$ that

$$B \cap A_n \in \{\emptyset, A_n\} \quad \forall n \in \mathbb{N}.$$

Show that $\mathcal{B} \subset \mathcal{A}$.

Problem 2 (weight 10%)

Let $p \in [1, \infty)$ and consider the normed vector space over $\mathbb{R}$ defined by

$$\ell^p(\mathbb{N}) = \left\{ x = (x_n)_{n \in \mathbb{N}} \subset \mathbb{R} \mid \|x\|_p := \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} < \infty \right\}.$$

Show that the norm $\|\cdot\|_p$ on $\ell^p(\mathbb{N})$ **cannot** be induced by an inner product unless $p = 2$.

Hint: You may use that fact that the *parallelogram law* must be satisfied for any inner-product induced norm.

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Problem 3

Consider the linear mapping $T : L^1([0, 1], \mathcal{B}([0, 1]), \lambda) \rightarrow \mathbb{R}$ defined by

$$T(f) := \int_{[0,1]} \int_{[0,1]} K(x, y) f(y) \lambda(dy) \lambda(dx)$$

where $K \in L^\infty([0, 1] \times [0, 1], \mathcal{B}([0, 1]) \otimes \mathcal{B}([0, 1]), \lambda \otimes \lambda)$ and $\mathcal{B}([0, 1]) = \mathcal{B}(\mathbb{R}) \cap [0, 1]$. For $p \in \{1, \infty\}$, we will below use the shorthand notation $L^p([0, 1]) := L^p([0, 1], \mathcal{B}([0, 1]), \lambda)$ and

$$L^p([0, 1] \times [0, 1]) := L^p([0, 1] \times [0, 1], \mathcal{B}([0, 1]) \otimes \mathcal{B}([0, 1]), \lambda \otimes \lambda).$$

3a (weight 10%)

Show that $h(x, y) := K(x, y)f(y)$ belongs to $L^1([0, 1] \times [0, 1])$ and explain why this implies that

$$T(f) = \int_{[0,1] \times [0,1]} K(x, y) f(y) \lambda \otimes \lambda(dx \times dy).$$

You may use without proof that the product of two measurable functions is a measurable function.

Hint: Show first that $\tilde{f}(x, y) := f(y)$ satisfies $\tilde{f} \in L^1([0, 1] \times [0, 1])$.

3b (weight 10%)

Find a constant $c > 0$ such that

$$|T(f)| \leq c \|f\|_{L^1([0,1])} \quad \forall f \in L^1([0, 1]),$$

and **explain** why there exists a $g \in L^\infty([0, 1])$ such that

$$T(f) = \int_{[0,1]} g(y) f(y) \lambda(dy) \quad \forall f \in L^1([0, 1]).$$

You may use without giving a proof that T is a linear mapping.

Problem 4

Given the measure space $((0, \infty), \mathcal{B}((0, \infty)), \lambda)$, where $\mathcal{B}((0, \infty)) := \mathcal{B}(\mathbb{R}) \cap (0, \infty)$, we consider the integral

$$I = \int_{(0, \infty)} x \lfloor x^{-1} \rfloor \lambda(dx),$$

where $\lfloor y \rfloor := \sup\{z \in \mathbb{Z} \mid z \leq y\}$ for $y \in \mathbb{R}$.

4a (weight 10%)

Show that $f : (0, \infty) \rightarrow \mathbb{R}$ defined by $f(x) = x \lfloor x^{-1} \rfloor$ is a $\mathcal{B}((0, \infty))/\mathcal{B}(\mathbb{R})$ -measurable function.

You may use without giving a proof that the product of two measurable functions is a measurable function.

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4b (weight 10%)

Compute the exact value of the integral I .

Hint: $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6$.

Problem 5 (weight 10%)

Compute the Fourier transform of the function $f(x) = \mathbb{1}_{[-1,1]}(x)$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ and use the result to show that

$$\int_{\mathbb{R}} \operatorname{sinc}^2(\xi) d\xi = \pi,$$

where

$$\operatorname{sinc}(x) := \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

Problem 6

Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where λ denotes the Lebesgue measure.

6a (weight 5%)

Formulate what it means for a measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ to be absolutely continuous with respect to λ .

6b (weight 15%)

Consider the Borel-measurable mapping $T : \mathbb{R} \rightarrow \mathbb{R}$ defined by $T(x) = x^2$ and the pushforward measure $\mu(B) := \lambda(T^{-1}(B))$ for $B \in \mathcal{B}(\mathbb{R})$. Show that the Radon–Nikodym derivative of μ with respect to λ is given by

$$\frac{d\mu}{d\lambda}(x) = \mathbb{1}_{(0,\infty)}(x) \frac{1}{\sqrt{x}}.$$

Hint: You may use without proof that $\mu([a, b]) = 2(\sqrt{b \vee 0} - \sqrt{a \vee 0})$ for all $[a, b] \in \mathbb{R}$ with $a \leq b$, when using the convention that $[a, b] = \emptyset$ when $a = b$.

THE END