

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT4400 — Linear Analysis with Applications

Day of examination: Friday, June 7, 2024

Examination hours: 15:00–19:00

This problem set consists of 2 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All subproblems (1a, 1b,...) carry the same weight.

Problem 1

1a

Formulate what it means that a measure is absolutely continuous with respect to another measure. Formulate the Radon–Nikodym derivative theorem.

1b

Consider the set $\mathbb{N}$ of natural numbers. Take two sequences $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ of strictly positive numbers and consider two measures μ and ν on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ (where $\mathcal{P}(\mathbb{N})$ denotes the σ -algebra of all subsets of $\mathbb{N}$) defined by

$$\mu(A) = \sum_{n \in A} a_n, \quad \nu(A) = \sum_{n \in A} b_n.$$

What is the Radon–Nikodym derivative $\frac{d\mu}{d\nu}$? Justify the answer.

Problem 2

Determine whether the following sequences $(f_n)_{n=1}^\infty$ converge in $L^1(0, 1)$ (we consider the Lebesgue measure on $(0, 1)$), and whenever they do, find the limits. Justify your answer.

2a

$f_n(x) = n\mathbb{1}_{(0, n^{-1}]}(x)$, where $\mathbb{1}_A$ denotes the characteristic function of a set A .

2b

$$f_n(x) = \frac{\sin x^n}{x^n}.$$

(Continued on page 2.)

Problem 3

3a

Formulate the Tonelli theorem.

3b

Assume $(X, \mathcal{B}, \mu)$ is a σ -finite measure space and $f: X \rightarrow [0, +\infty)$ is a measurable function. For every $t \geq 0$ consider the set $A_t = \{x \in X : f(x) \geq t\}$. Show that

$$\int_X f \, d\mu = \int_{[0, +\infty)} \mu(A_t) \, d\lambda(t),$$

where λ denotes the Lebesgue measure. Hint: integrate the characteristic function of the set $A = \{(t, x) : f(x) \geq t\}$.

3c

In the setting of (b), show that for every number $p > 0$ we have

$$\int_X f^p \, d\mu = p \int_{[0, +\infty)} t^{p-1} \mu(A_t) \, d\lambda(t).$$

Problem 4

Consider the space $\mathcal{S}(\mathbb{R})$ of Schwartz functions on $\mathbb{R}$. Show using the Fourier transform that for every $g \in \mathcal{S}(\mathbb{R})$ there exists a unique function $f \in \mathcal{S}(\mathbb{R})$ such that

$$f'' - f = g.$$

Problem 5

Assume $f: \mathbb{R} \rightarrow [0, +\infty)$ is a Lebesgue measurable integrable (with respect to the Lebesgue measure) function. For $x \in \mathbb{R}$, define

$$g(x) = \liminf_{n \rightarrow +\infty} f(x + n).$$

5a

Show that the function g is integrable.

5b

Show that g is periodic and conclude that $g = 0$ a.e.

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