

• digression on entanglement, partial measurement
(§ 10.1.2)

consider three-qubit system $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ Alice Bob Charlie.

and the Greenberger-Horne-Zeilinger state

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|100\rangle + |111\rangle)$$

this is an entangled state.

(i.e. $|\psi\rangle \neq |\phi_1\rangle \otimes |\phi_2\rangle \otimes |\phi_3\rangle$ for $|\phi_i\rangle \in \mathbb{C}^2$)

Q. how does this state look to Alice & Bob?

~> we should look at partial measurement

i.e. first compute $P_\psi = |\psi\rangle\langle\psi|$, then

$$\text{take } P_\psi^{AB} = (\text{id} \otimes \text{id} \otimes \text{Tr})(P_\psi)$$

$$= \sum_{i=0,1} (I_2 \otimes I_2 \otimes \langle i|) P_\psi (I_2 \otimes I_2 \otimes |i\rangle)$$

$$P_\psi = \frac{1}{2} (|100\rangle\langle 000| + |100\rangle\langle 111| + |111\rangle\langle 000| + |111\rangle\langle 111|)$$

$$\Rightarrow P_\psi^{AB} = \frac{1}{2} (|00\rangle\langle 00| + |11\rangle\langle 11|)$$

this is a convex combination of $P_{00} = |00\rangle\langle 00|$ and $P_{11} = |11\rangle\langle 11|$

i.e. mix. of unentangled states.

So the measurements of Alice and Bob will be correlated but not "quantum" effect from entanglement (if they use $|0\rangle, |1\rangle$).

Compare: max ent. stt $|\phi\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$

$$\text{will give } P_\phi = \frac{1}{2} (|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)$$

So we can say:

Def a density mat ρ on $\mathbb{C}^{N_A} \otimes \mathbb{C}^{N_B}$ represents

a separable state if $\rho = \sum_{i \text{ s.t. } p_i} p_i (|\psi_i\rangle \otimes |\psi'_i\rangle) (\langle\psi_i| \otimes \langle\psi'_i|)$

for some states $|\psi_i\rangle \in \mathbb{C}^{N_A}$, $|\psi'_i\rangle \in \mathbb{C}^{N_B}$, $p_i \geq 0$, $\sum p_i = 1$.

(cont.) P represents an entangled state
if it's not of this form.

- Quantifying entanglement by entropy (Lect 11)

• "classical" formulation of entropy:

probability distribution $P = (p_i)_{0 \leq i < n}$

$$(p_i \geq 0, \sum p_i = 1) \quad (0 \log 0 = 0)$$

$\Rightarrow$ Shannon entropy $H(P) = - \sum_{0 \leq i < n} p_i \log_2(p_i)$

this quantifies the amount of information carried by one digit
inside a random string $l_0 l_1 \dots$, $0 \leq l_k < n$

Ex. uniform distr. $p_i = \frac{1}{n} \Rightarrow H(P) = \log_2 n$

if $p_0 = 1, p_i = 0 \ (i > 0) \Rightarrow H(P) = \log_2 1 = 0$

Another convention: natural log. instead of $\log_2$

(then everything is scaled by $\log 2$)

• "Quantum" entropy (§ 11.2)

Def. ρ : density matrix on $\mathbb{C}^N$.

the von Neumann entropy of ρ is

$$S(\rho) = - \sum_{0 \leq i < N} \gamma_i \log_2 \gamma_i \quad ; \quad (\gamma_i): \text{eigenvals of } \rho$$

So $S(\text{diag}(P)) = H(P)$ for prob. distr. P .

Rem / Fact

ρ density mat $\Rightarrow$ diagonal entries give a prob. distr.

$$\text{write } \text{diag}(\rho) = \begin{bmatrix} p_{00} & 0 \\ 0 & p_{n-1, n-1} \end{bmatrix},$$

$$H(\text{diag}(\rho)) = S(\text{diag}(\rho)) \geq S(\rho)$$

"concavity of entropy"



$$\Rightarrow S(\rho) = \min_{U \in U(n)} H(\text{diag}(U \rho U^\dagger))$$

• entanglement entropy (§ 11.2.1)

$$|\psi\rangle \in \mathbb{C}^{N_A} \otimes \mathbb{C}^{N_B}, \quad \rho_\psi = |\psi\rangle\langle\psi|$$

$$\rho^A = (\text{id} \otimes \text{Tr})(\rho), \quad \rho^B = (\text{Tr} \otimes \text{id})(\rho)$$

Thm. $S(\rho^A) = S(\rho^B)$

Key point: the Schmidt decomposition

$$|\psi\rangle = \sum_{0 \leq i < k} \lambda_i |v_i\rangle \otimes |w_i\rangle \quad (\text{also write } \sum \lambda_i |v_i\rangle |w_i\rangle)$$

such that: $(|v_i\rangle)_i$ orthon. system in $\mathbb{C}^{N_A}$
 $(|w_i\rangle)_i$ $\sim$ $\mathbb{C}^{N_B}$

Why this helps:

add $|v_k\rangle, \dots, |v_{N_A-1}\rangle, |w_k\rangle, \dots, |w_{N_B-1}\rangle$ to
 get an orthonormal basis of $\mathbb{C}^{N_A}, \mathbb{C}^{N_B}$.

$$\text{Tr}(S) = \sum (v_i | S | v_i) \quad \text{for } S \text{ on } \mathbb{C}^{N_A}$$

$$\text{Tr}(T) = \sum (w_i | T | w_i) \quad \text{for } T \text{ on } \mathbb{C}^{N_B}$$

$$\text{So } \rho^A = \sum_{0 \leq i < k} \lambda_i^2 |v_i\rangle\langle v_i|, \quad \rho^B = \sum_{0 \leq i < k} \lambda_i^2 |w_i\rangle\langle w_i|$$

$\Rightarrow$ they have the same eigenvals $(\lambda_i)_i$

$$\Rightarrow S(\rho^A) = -\sum \lambda_i^2 \log_2 \lambda_i^2 = S(\rho^B)$$

How to get the Schmidt decomp:

$$\text{write } |\psi\rangle = \sum_{0 \leq j < N_A} \sum_{0 \leq j' < N_B} \alpha_{jj'} |j\rangle |j'\rangle$$

$$\text{and } A = [\alpha_{jj'}]_{j,j'} \in M_{N_A \times N_B}. \quad (A = \sum \alpha_{jj'} |j\rangle\langle j'|)$$

$\leadsto A^\dagger A \in M_{N_B}$ is pos. semidef.

$|w_i'\rangle$: eigenvectors, eigenval $\sqrt{\lambda_i}$ ($\lambda_i > 0$)

$$\lambda_i |v_i\rangle = A |w_i'\rangle \quad \text{i.e. } A = \sum \lambda_i |v_i\rangle\langle w_i'| \quad \text{see next p.}$$

Def. $E(|\psi\rangle) = S(\rho^A) = S(\rho^B)$ entanglement entropy
 of $|\psi\rangle \in \mathbb{C}^{N_A} \otimes \mathbb{C}^{N_B}$

Ex. $|\psi\rangle = |\varphi_1\rangle|\varphi_2\rangle$ separable

$$\rho^A = |\varphi_1\rangle\langle\varphi_1| \quad E(|\psi\rangle) = S(\rho^A) = 0.$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \quad \text{max. ent.}$$

$$\rho^A = \frac{1}{2} I_2 \quad E(|\psi\rangle) = \log_2(2) = 1.$$

generally: $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ would give

ρ^A : 2×2 pos. sem. def. $\rightarrow$ eigenval $\lambda, 1-\lambda$

$$E(|\psi\rangle) = -(\lambda \log_2 \lambda + (1-\lambda) \log_2 (1-\lambda)) \leq 1. \quad \text{graph of } -x \log x$$

... for density matrix ρ on $\mathbb{C}^2 \otimes \mathbb{C}^2$.

analogues of $E(|\psi\rangle)$ via the interpretation

$$E(|\psi\rangle) \sim \frac{\# \text{ of } |\psi\rangle \text{ "in" } |\psi\rangle \dots |\psi\rangle}{k} \quad \text{with } k \text{ terms}$$

$|w_i\rangle$ from $|w'_i\rangle$:

$$\text{write } |w'_i\rangle = \sum_{0 \leq j < N_B} \beta_{ij} |j\rangle, \quad |w_i\rangle = \sum \beta_{ij}^* |j\rangle$$

$$\begin{cases} A = \sum \alpha_{ij} |i\rangle\langle j| \\ A = \sum_i \lambda_i |v_i\rangle\langle w'_i| = \sum_{i,j} \lambda_i \beta_{ij}^* |v_i\rangle\langle j| \end{cases}$$

$$\Rightarrow \sum \alpha_{ij} |i\rangle\langle j| = \sum \lambda_i \beta_{ij}^* |v_i\rangle\langle j| \\ = \sum \lambda_i |v_i\rangle\langle w_i|$$